

Report 4622

METHODS FOR BOUNDARY-VALUE PROBLEMS IN FREE-SURFACE FLOWS
THE THIRD DAVID W. TAYLOR LECTURE - 27 AUGUST THROUGH 19 SEPTEMBER 1974

AD A093352

LEVEL II

**DAVID W. TAYLOR NAVAL SHIP
RESEARCH AND DEVELOPMENT CENTER**

Bethesda, Md. 20084



**METHODS FOR BOUNDARY-VALUE PROBLEMS IN
FREE-SURFACE FLOWS**

THE THIRD

DAVID W. TAYLOR LECTURE

27 AUGUST THROUGH 19 SEPTEMBER 1974

by

**Professor John V. Wehausen
The University of California
Berkeley**

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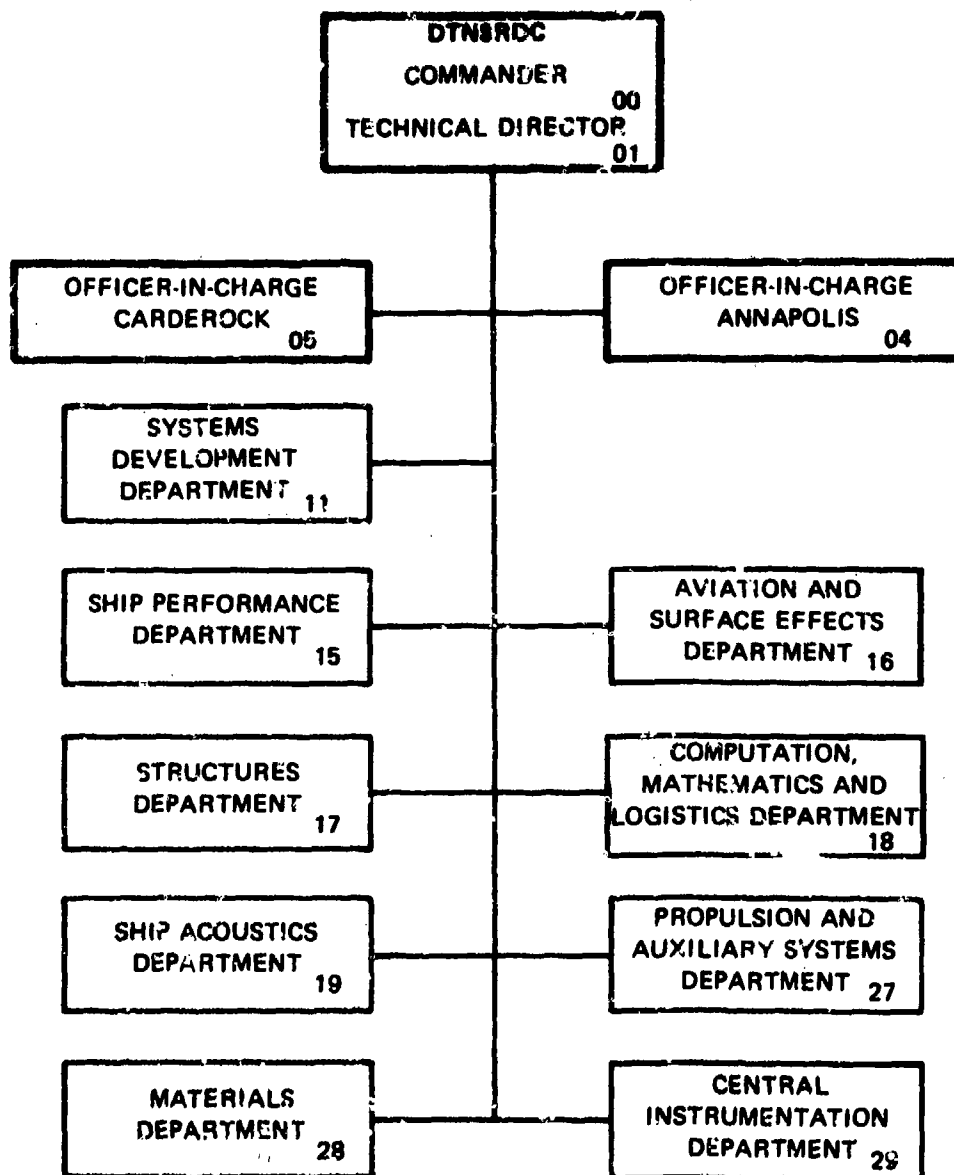
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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 4622 ✓	2. GOVT ACCESSION NO. AD-A093 352	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) METHODS FOR BOUNDARY-VALUE PROBLEMS IN FREE-SURFACE FLOWS: THE THIRI DAVID W. TAYLOR LECTURE		5. TYPE OF REPORT & PERIOD COVERED
6. AUTHOR(s) Professor John V. Wehausen (The University of California, Berkley)		7. PERFORMING ORG. REPORT NUMBER
8. PERFORMING ORGANIZATION NAME AND ADDRESS (12) 232/		9. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit 4-1500-001-41/1509
10. CONTROLLING OFFICE NAME AND ADDRESS David W. Taylor Naval Ship Research and Development Center Bethesda, Md. 20084		11. REPORT DATE September 1974
12. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (14) DTN5RDZ-4622 ✓		13. NUMBER OF PAGES 125
14. DISTRIBUTION STATEMENT (of this Report) Approved for Public Release: Distribution Unlimited		15. SECURITY CLASS. (of this report) UNCLASSIFIED
15. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		16. DECLASSIFICATION/DOWNGRADING SCHEDULE
17. SUPPLEMENTARY NOTES		
18. KEY WORDS (Continue on reverse side if necessary and identify by block number) Boundary-Value Problems Green Functions Free-Surface Flows Multipoles Fourier Method Reduction Method		
19. ABSTRACT (Continue on reverse side if necessary and identify by block number) In these lectures several methods are presented that are useful for solving free-surface boundary-value problems: separation of variables and the Fourier method; the method of reduction and reflection; the method of Green functions; and the method of multipoles. Each method is illustrated by one or more examples. In the examples the fluid is assumed to be inviscid and incompressible and the flow irrotational. The boundary conditions have (Continued on reverse side)		

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TABLE OF CONTENTS

	Page
ABSTRACT	1
INTRODUCTION	1
PART 1: THE BASIC EQUATIONS	4
PART 2: THE FOURIER METHOD	7
PART 3: THE REDUCTION METHOD	35
PART 4: THE METHOD OF GREEN FUNCTIONS	59
PART 5: THE METHOD OF MULTIPOLES	90
BIBLIOGRAPHY	114

LIST OF FIGURES

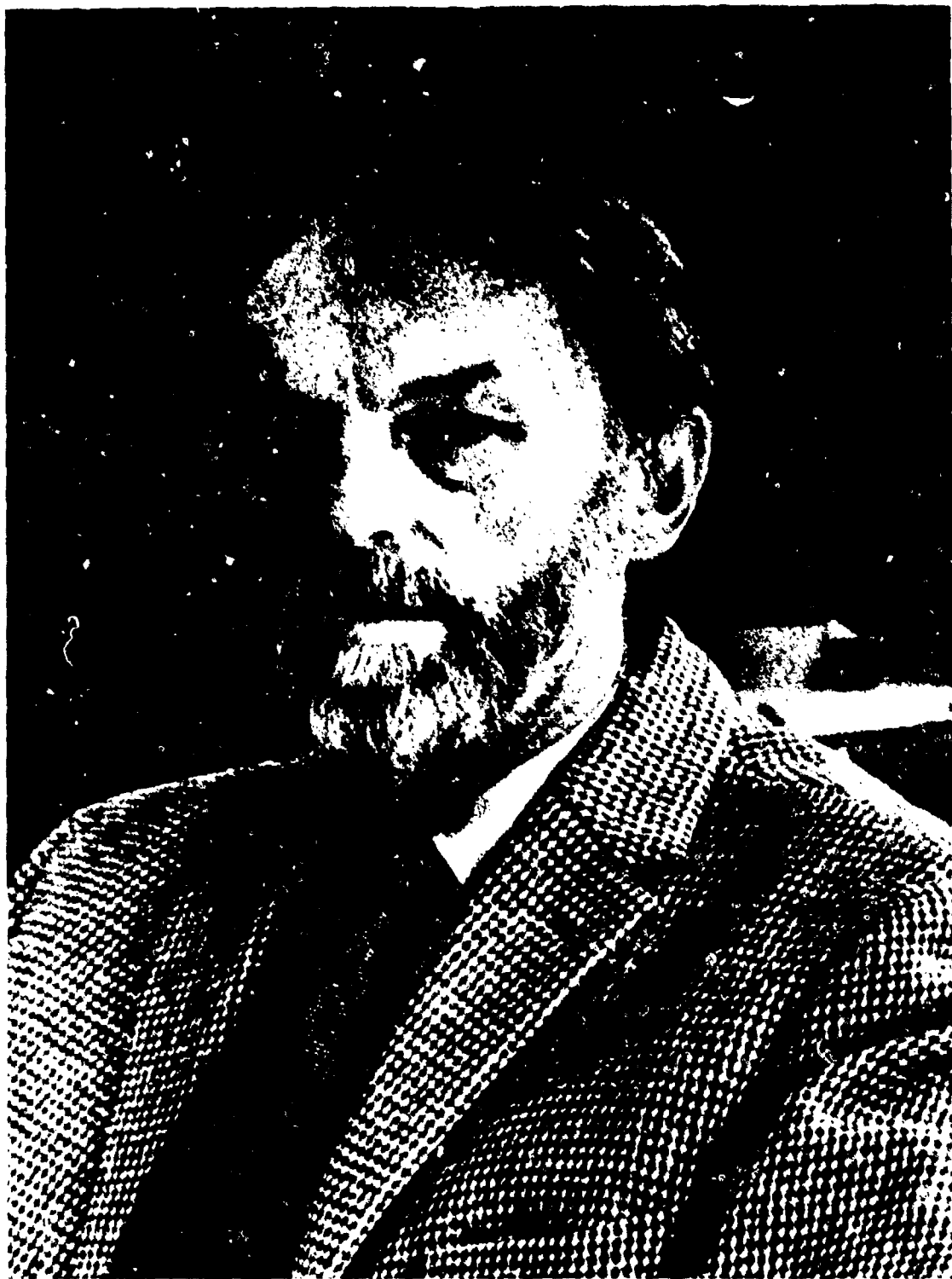
1 - Graphic Solution of First Equation in (2.16)	11
2 - Graphic Solution of Second Equation in (2.16)	12
3 - Critical Frequencies for Wavemaker in a Channel	24
4 - Representation of a Sloshing Progressive Wave	25
5 - Representation of Third Type of Progressive Wave	26
6 - Flat Plate in Oncoming Waves	37
7 - Path of Integration	41
8 - Region for Laurent Series	44
9 - Path of Integration for Equation (3.32)	47
10 - Deformed Path of Integration	50
11 - Path of Integration below the Plate	51
12 - Completion of Path of Integration	51

	Page
13 - Path of Integration around Extended Flat Plate	52
14 - R , T , and β_R as Functions of k	57
15 - Representation of the Problem of Forced Harmonic Motion	62
16 - Limitation on the Part of the Bottom Not at Depth h	71
17 - Representation of the Physical Problem of Diffraction about Islands	75
18 - Reference Axes for Diffraction about Islands	75
19 - Region S for Helmholtz Equation	77
20 - Region Bounded by C and Σ_R	78
21 - Incident Wave on a Coastline with Harbor	80
22 - Straight-Line Coast	80
23 - Circular Coast	81
24 - Oscillating Body on the Free Surface	90
25 - Region of Convergence	92
26 - Path of Integration for (5.12)	93
27 - Paths of Integration in ζ -Plane	96
28 - Transformed Paths in the k -Plane	97
29 - Arcs Σ_R Joining the Path to ∞	97
30 - Path along the x -Axis	98
31 - Equivalent Path of Integration for $f(z)$	99
32 - Definition of Angle γ	102
33 - ζ - and z -Planes	105

PREFACE

The David W. Taylor Lectures were initiated as a living memorial to our founder, in recognition of his many contributions to naval architecture and naval hydrodynamics. Admiral Taylor was a pioneer in the use of hydrodynamic theory and mathematics for the solution of naval problems. The system of mathematical lines developed by Taylor was used to develop many ship designs for the Navy long before the computer was invented. He founded and directed the Experimental Model Basin; perhaps most important of all, he established a tradition of applied scientific research at the "Model Basin" which has been carefully nurtured through the decades and which we treasure and protect today. In the spirit of this tradition, we invite an eminent scientist in a field closely related to the Center's work to spend a few weeks with us, to consult with and advise our working staff, and to give a series of lectures on subjects of current interest.

Our third lecturer in this series is Professor John V. Wehausen of the University of California, Berkeley. Professor Wehausen graduated from the University of Michigan and received his Ph.D. in mathematics from there in 1938. He is no stranger to the Center or its problems. From 1946 to 1949 he worked here as a mathematician and developed his interest in naval hydrodynamics. He later became the resident mathematician in the Department of Naval Architecture at the University of California, where he further developed his knowledge of the theory of water waves and the hydrodynamics of ships. Professor Wehausen has trained many outstanding researchers in the field of naval hydrodynamics through his teaching at the University, and we are most honored that he agreed to be a David W. Taylor lecturer.





ABSTRACT

In these lectures several methods are presented that are useful for solving free-surface boundary-value problems: separation of variables and the Fourier method, the method of reduction and reflection, the method of Green functions, and the method of multipoles. Each method is illustrated by one or more examples. In the examples the fluid is assumed to be inviscid and incompressible and the flow irrotational. The boundary conditions have been linearized. The examples themselves are all concerned with diffraction and forced motion. Although the methods are applicable to a much wider class of problems, this restriction allows a simple formulation of the physical problem and immediate involvement with the method itself.

INTRODUCTION

The analysis of an engineering problem in fluid mechanics usually proceeds along the following lines. First one selects a model of a fluid: Navier-Stokes or inviscid, compressible or not, with or without surface tension, etc. The decision can usually be based upon an examination of various physical parameters characterizing the problem. Next a decision must be made with regard to the flow: is it steady or unsteady, is it irrotational, etc.? Again, this will be based upon physical parameters (or perhaps a desperate need to avoid complications). The next step is an exact formulation of the equations of motion and of the boundary and initial conditions for the fluids and bodies present. (Ideally this should be accompanied by existence and uniqueness studies.) The problem will usually be too difficult mathematically, so the next step is to replace these equations by other simpler ones. The region of usefulness of these simpler equations will also depend on certain parameters associated with the problem. Finally, assuming that the equations are now tractable, one proceeds to solve them either analytically or perhaps by direct numerical approximation methods.

One must grant immediately that the distinction between these steps is not always clear. Is approximating the Navier-Stokes equations by the Euler equations really different from approximating the flow about an elongated body by that given by a slender-body approximation? One might argue that the first involves a physical parameter associated with the fluid and the latter a geometrical parameter, but both are approximations. As further cases are considered, one realizes that the boundaries are fuzzy.

However, these problems, important and fascinating as they are, will not concern us here for we are going to limit ourselves to just one aspect of the procedure outlined above, namely, the finding of analytical solutions once a tractable problem has been formulated. Indeed, we shall restrict our attention still further, namely, to problems with a free surface in which the fluid is assumed to be inviscid, incompressible, and subject to a gravitational force and in which the flow is assumed to be irrotational. In addition to these restrictions we shall also assume that the boundary conditions are linearized.

It is obvious that we are bypassing a host of interesting questions. We shall not consider how or whether the linearized equations we deal with can be made part of a systematic approximation scheme. This problem will be relegated to one of the other "steps" described above. We have restricted the purview of these lectures so drastically for the following reasons. Although a single topic could be selected and developed fairly intensively, in a set of eight lectures each lecture would have to be built upon the preceding ones. The audience would necessarily have to devote some thought between lectures to the topic under discussion in order to keep the whole development in mind. And, of course, regular attendance would be necessary. Since the situation is different from a university, where both outside study and regular attendance can be expected, it seemed more useful to select a subject that can easily be divided into one- or two-lecture units so that not too much baggage has to be accumulated as one proceeds. I believe and hope that this will not make the lectures less interesting.

The lectures will consist of a discussion of the several methods in use for finding analytical solutions for free-surface problems. Each method will be illustrated by one or more examples, which we hope will themselves have some intrinsic interest. Several of the examples could be solved by more than one of the methods, and it would undoubtedly be interesting to do this. The original intention was not to do so in order to obtain a greater diversity of physical problems. However, the examples treated do tend to fall into one area: diffraction and forced harmonic motion. This was not necessary. It just so happened that such problems seemed to be among the simplest with which to illustrate the various methods.

The methods will be chiefly the following: separation of variables and the Fourier method, the method of reduction and reflection, the method of Green functions, the method of multipole expansions, and, time permitting,* variational methods. I shall try to indicate some of the advantages and disadvantages of each and the limitations in their use. Since the easiest problems have usually been solved first, the examples will often not be from the recent literature. On the other hand, they are also the most useful for purposes of illustration just because they are fairly simple. I shall try to compensate for this by calling attention to recent literature in which the method under discussion has been used. I do not wish to imply that solving more complex problems simply required the investigators to turn the crank a little harder. Difficulties almost always arise in applying one of the methods to a new problem, and it may be necessary to alter it in some appropriate way.

*It did not.

PART 1: THE BASIC EQUATIONS

As coordinate system we begin with an inertial one fixed in the fluid. We choose it rectangular and right-handed and with Oy directed oppositely to the force of gravity in both two and three dimensions. The undisturbed water surface will be taken as the (x,z) plane. As stated above, we suppose the fluid to be inviscid and incompressible and the motion irrotational. Consequently, there exists a velocity potential $\Phi(x,y,z,t)$ with absolute velocity

$$V = \text{grad } \Phi = \nabla \Phi = (\Phi_x, \Phi_y, \Phi_z) \quad (1.1)$$

Conservation of mass gives

$$\Delta \Phi = \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \quad (1.2)$$

The three conservation-of-momentum equations (Euler equations) reduce to Euler integral

$$\Phi_t + \frac{1}{2} |\nabla \Phi|^2 + gy + p/\rho = \text{constant} \quad (1.3)$$

If the free surface is represented by

$$y = Y(x,z,t)$$

then the kinematic boundary condition on the free surface is

$$\Phi_x(x,Y,z,t)Y_x - \Phi_y + \Phi_z Y_z + Y_t = 0 \quad (1.4)$$

and the dynamic boundary condition is

$$\Phi_t(x,Y,z,t) + \frac{1}{2} |\nabla \Phi|^2 + gY = 0 \quad (1.5)$$

If a solid boundary is present, then the kinematic boundary condition states that

$$\phi_n = \vec{V} \cdot \vec{n} \quad (1.6)$$

at each point of the surface, where \vec{V} is its velocity at that point. If the surface is given by

$$F(x, y, z, t) = 0 \quad (1.7)$$

this condition may be written in the form

$$\phi_x F_x + \phi_y F_y + \phi_z F_z + F_t = 0 \quad (1.8)$$

Dynamic boundary conditions may also be necessary on solid boundaries. For example, in the case of freely floating bodies, these take the form of the Newton equations for motion of a rigid body.

We have stated earlier that we shall consider only linearized problems without discussing the rationale of the linearization. The Laplace equation is already linear, but the solution of a nonlinear problem will not, of course, be the same as that of a corresponding linearized problem. The two free-surface conditions become

$$\phi_y(x, 0, z, t) - Y_t = 0 \quad (1.9)$$

$$\phi_t(x, 0, z, t) + gY = 0 \quad (1.10)$$

If Y is eliminated, this condition becomes

$$\phi_{tt}(x, 0, z, t) + g\phi_y = 0 \quad (1.11)$$

Linearization of the boundary conditions on solid boundaries is more complicated for it will depend upon the nature of the problem and the basis of the linearization.

The free-surface condition (1.11) may appear in different forms, depending upon the circumstances. If the motion is assumed to be harmonic in time with frequency σ , we may write

$$\begin{aligned}\phi(x,y,z,t) &= \phi_1(x,y,z) \cos \sigma t + \phi_2(x,y,z) \sin \sigma t \\ &= R_e [\phi e^{-i\sigma t}], \quad \phi = \phi_1 + i\phi_2\end{aligned}\tag{1.12}$$

Then (1.11) becomes

$$\phi(x,0,z) - \frac{g}{\sigma^2} \phi_y = 0\tag{1.13}$$

If it is convenient to use a moving coordinate system $x = \bar{x} + ct$, $y = \bar{y}$, $z = \bar{z}$, and if $\phi(\bar{x}, \bar{y}, \bar{z}, t) = \phi(x, y, z, t)$ then (1.11) becomes

$$c^2 \phi_{\bar{x}\bar{x}}(\bar{x}, 0, \bar{z}, t) - 2c\phi_{\bar{x}t} + \phi_{tt} - \dot{c}\phi_{\bar{x}} + g\phi_{\bar{y}} = 0$$

If the motion is steady in the moving coordinate system, then ϕ does not depend on t and the three middle terms drop out.

We assume that these equations and their provenance is, infact, known. They are given only for ready reference.

PART 2: THE FOURIER METHOD

This is probably the oldest and most widely used method for solving boundary-value problems in mathematical physics. It is the classical method of separation of variables, determination of eigenfunctions, and expansion of the solution in them. The expression may take the form of either a sum or an integral, and we shall give an example of each.

A limitation of the method in application results from the fact that we usually would like to identify the surface upon which boundary values are imposed with coordinate surfaces of the variables. This explains why so many of the examples of classical physics deal with rectangular, circular, and elliptical cylinders and with rectangular solids, spheres, ellipsoids, etc. These all fit into some system of variables in which the equations are separable. However, the possibilities are limited for there are only eleven coordinate systems in which the three-dimensional Laplace equation can be separated and five in which the two-dimensional one can be separated (see e.g., an article by N. Levinson, B. Bogert, and R.M. Redheffer on pages 241-262 of Volume 7 (1949) of the Quarterly of Applied Mathematics and references cited there). We shall be concerned only with coordinate systems in which one of the coordinates is y .

SEPARATION OF VARIABLES

We shall look for a solution in the form

$$\Phi(x,y,z,t) = \chi(x,z) Y(y) T(t) \quad (2.1)$$

Substitution into $\Delta\Phi = 0$ yields

$$(\chi_{xx} + \chi_{zz}) YT + \chi Y''T = 0$$

or, if we assume χ and Y are not identically zero,

$$\frac{\Delta_2 \chi}{\chi} - \frac{Y''}{Y} = 0 \quad (2.2)$$

Each term on the left-hand side must be a constant, say, α and β , respectively, where

$$\alpha + \beta = 0 \quad (2.3)$$

Hence

$$\Delta_2 \chi - \alpha \chi = 0, \quad Y'' - \beta Y = 0 \quad (2.4)$$

It was stated above that one of the situations in which application of separation of variables is most convenient is where one of the boundaries is a coordinate surface. The surface $y = 0$ is already such a surface. Let us choose the bottom to be another such. We shall consider two cases. In one, a horizontal bottom exists at $y = -h$. In the other, the fluid is infinitely deep. In the first case, the boundary condition (1.6) becomes

$$\phi_y(x, -h, z, t) = 0 \quad (2.5a)$$

In the second case, we shall relax this to the condition

$$|\phi_y(x, y, z, t)| < M < \infty \text{ as } y \rightarrow -\infty \quad (2.5b)$$

For the function form (2.1), these imply

$$Y'(-h) = 0 \text{ and } |Y'(y)| < N < \infty \text{ as } y \rightarrow -\infty \quad (2.6)$$

We next substitute (2.1) into the free-surface condition (1.11) to obtain

$$\chi(x, z)Y(0)T''(t) + g\chi Y'T = 0$$

or

$$T''(t) + g \frac{Y'(0)}{Y(0)} T = 0$$

We shall restrict attention to the case where the coefficient of T is positive and write

$$\sigma^2 = gY'(0)/Y(0) \quad (2.7)$$

The solution for T is then

$$T = A \cos \sigma t + B \sin \sigma t \quad (2.8)$$

Let us now consider the equation for Y ,

$$Y'' - \beta Y = 0$$

If we choose $\beta = k^2 > 0$, then

$$Y = C e^{ky} + D e^{-ky}$$

The boundary conditions in (2.6) then yield

$$Y = \cosh k(y + h) \text{ and } Y = e^{ky} \quad (2.9)$$

respectively. (The multiplicative constant can be safely dropped since this is already provided for in T in (2.8).)

But now (2.7) gives

$$\sigma^2 = gk \text{ and } \sigma^2 = gk \tanh kh \quad (2.10)$$

for $h = \infty$ and $h < \infty$, respectively.

From (2.3) and (2.4), we then find that χ must satisfy

$$\Delta_2 \chi + k^2 \chi = 0 \quad (2.11)$$

Next we choose $\beta = -k^2 < 0$. Then

$$Y = C \cos ky + D \sin ky$$

The boundary condition $Y'(-h) = 0$ yields

$$Y = \cos k(y + h) \quad (2.12)$$

For infinite depth, boundedness imposes no further condition. However, from (2.7) we deduce

$$\sigma^2 = gk \frac{D}{C}$$

so that we may write

$$Y = \cos ky + \frac{\sigma^2}{gk} \sin ky \quad (2.13)$$

Equation (2.13) reflects only the effect of the free surface and not of the bottom.

Equation (2.12) substituted into (2.7) gives

$$\sigma^2 = -gk \tan kh \quad (2.14)$$

The equation for χ that is associated with $\beta = -k^2$ is

$$\Delta_2 \chi - k^2 \chi = 0 \quad (2.15)$$

Let us now suppose that σ is fixed and ask what values of k will be determined for $h < \infty$ by (2.10) and (2.14). This is easily seen from a graphical display. We write (2.10) and (2.14) in the form

$$\frac{\sigma^2 h}{g} \frac{1}{kh} = \tanh kh \text{ and } \frac{\sigma^2 h}{g} \frac{1}{kh} = -\tan kh \quad (2.16)$$

A graphical solution of the first equation shows a single solution (see Figure 1).

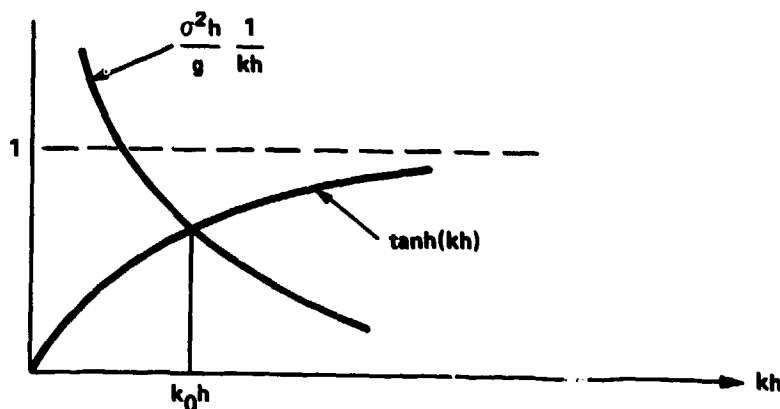


Figure 1 - Graphic Solution of First Equation in (2.16)

By the same method one finds an infinite number of solutions for the second equation (see Figure 2).

For the case $h < \infty$, we have now obtained the following solutions for ϕ :

$$\begin{aligned} \chi_0(x, z) \cosh k_0(y + h) [A_0 \cos \sigma t + B_0 \sin \sigma t] \\ \sigma^2 = gk_0 \tanh k_0 h, \Delta_2 \chi + k_0^2 \chi = 0 \end{aligned} \quad (2.17)$$

and

$$\chi_1(x, z) \cos k_1(y + h) [A_1 \cos \sigma t + B_1 \sin \sigma t] \quad (2.17)$$

$$\sigma^2 = -gk_1 \tan k_1 h, \Delta_2 \chi_1 - k_1^2 \chi_1 = 0$$

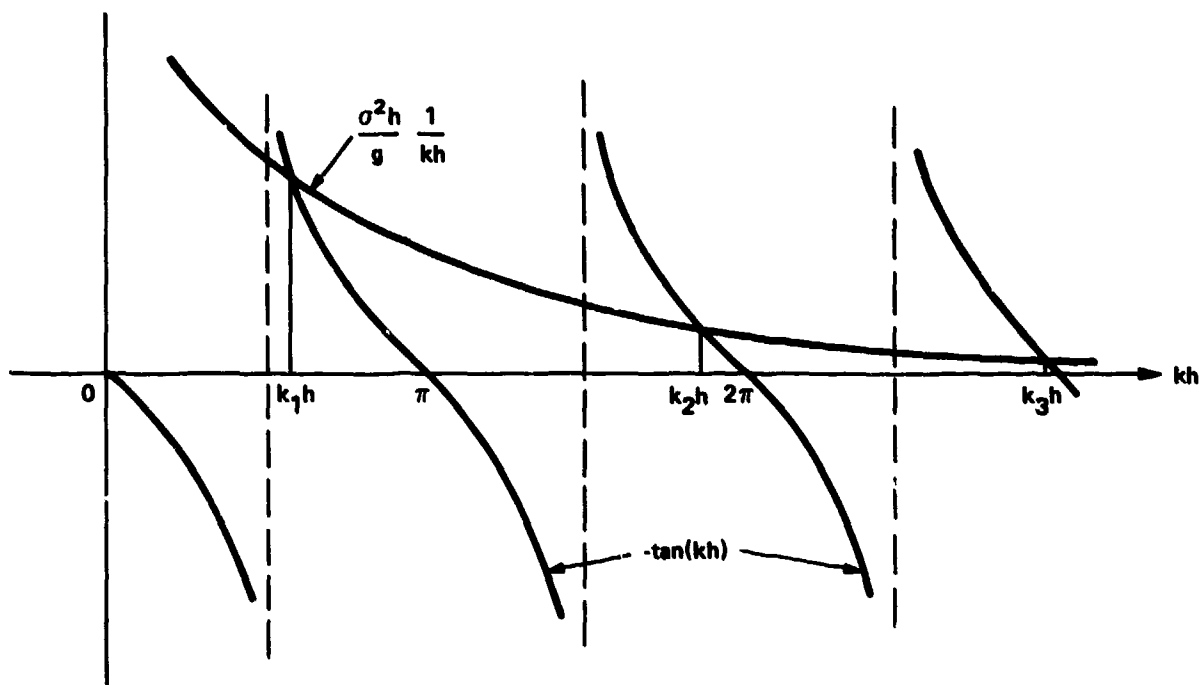


Figure 2 - Graphic Solution of Second Equation found in (2.16)

A sum of such terms (appropriately convergent) will evidently also be a solution. With regard to the family of functions

$$\{\cosh k_0(y + h), \cos k_1(y + h), \cos k_2(y + h) \dots\} \quad (2.18)$$

it is possible to prove the following property: Any pair of functions is orthogonal on the interval $(-h, 0)$, i.e.,

$$\int_{-h}^0 \cosh k_0(y+h) \cos k_1(y+h) dy = 0$$

$$\int_{-h}^0 \cos k_i(y+h) \cos k_j(y+h) dy = 0 \text{ if } i \neq j$$
(2.19)

One may prove also the following theorem:

Theorem: If $f(y)$ is defined and square-integrable on the interval $[-h, 0]$, then

$$f(y) = b_0 \cosh k_0(y+h) + \sum_{i=1}^{\infty} b_i \cos k_i(y+h)$$
(2.20)

where

$$b_0 = \frac{4k_0}{\sinh 2k_0 h + 2k_0 h} \int_{-h}^0 f(y) \cosh k_0(y+h) dy$$

and

$$b_i = \frac{4k_i}{\sin 2k_i h + 2k_i h} \int_{-h}^0 f(y) \cos k_i(y+h) dy$$

The formulas for the b 's follow from the orthogonality.

For $h = \infty$, we have the following solutions for ϕ :

$$\chi(x, z) e^{ky} [A \cos \sigma t + B \sin \sigma t],$$

$$\sigma^2 = gk, \Delta_2 \chi + k^2 \chi = 0$$

$$\chi(x, z) \left[\cos ky + \frac{\sigma^2}{gk} \sin ky \right] [A \cos \sigma t + B \sin \sigma t], \quad (2.21)$$

$$k \text{ arbitrary, } \Delta_2 \chi - k^2 \chi = 0$$

Do we have anything analogous to the representation theorem (2.20)? There is, in fact, an analog, first given by Havelock in 1929, that is reducible to the Fourier-integral theorem. It is as follows:

Theorem: If $f(y)$ is absolutely integrable on $(-\infty, 0)$, then

$$f(y) = \frac{2}{\pi} \int_0^\infty dk \int_{-\infty}^0 d\eta f(\eta) \frac{(k \cos ky + v \sin ky)(k \cos k\eta + v \sin k\eta)}{k^2 + v^2} \quad (2.22)$$

$$+ 2v e^{vy} \int_{-\infty}^0 f(\eta) e^{v\eta} d\eta, \quad v = \sigma^2/g$$

We may write this in a form more closely analogous to (2.20) by presenting it as a pair of transforms:

$$f(y) = b_0 e^{vy} + \int_0^\infty b(k) (k \cos ky + v \sin ky) dk$$

$$b_0 = 2v \int_{-\infty}^0 f(\eta) e^{v\eta} d\eta \quad (2.23)$$

$$b(k) = \frac{2}{\pi} \int_{-\infty}^0 f(\eta) \frac{k \cos k\eta + v \sin k\eta}{k^2 + v^2} d\eta$$

The proof of (2.22) is by reduction to the ordinary Fourier integral theorem.
It may be presented as follows:

$$\begin{aligned}
 f(y) &= \frac{2}{\pi} \int_0^{\infty} dk \int_{-\infty}^0 d\eta f(\eta) \left[\frac{k^2}{k^2 + v^2} \cos ky \cos k\eta + \frac{v^2}{k^2 + v^2} \sin ky \sin k\eta \right. \\
 &\quad \left. + \frac{kv}{k^2 + v^2} (\sin ky \cos k\eta + \cos ky \sin k\eta) \right] \\
 &\quad + 2v e^{vy} \int_{-\infty}^0 f(\eta) e^{v\eta} d\eta \\
 &= \frac{2}{\pi} \int_0^{\infty} dk \int_{-\infty}^0 d\eta f(\eta) \left[\cos ky \cos k\eta - \frac{v^2}{k^2 + v^2} \cos k(y + \eta) \right. \\
 &\quad \left. + \frac{kv}{k^2 + v^2} \sin k(y + \eta) \right] \\
 &\quad + 2v e^{vy} \int_{-\infty}^0 f(\eta) e^{v\eta} d\eta \\
 &= \frac{1}{\pi} \int_0^{\infty} dk \int_{-\infty}^0 d\eta f(\eta) [\cos k(y + \eta) + \cos k(y - \eta)] \\
 &\quad + \frac{2}{\pi} \int_{-\infty}^0 d\eta f(\eta) \left[-\frac{\pi}{2} \frac{v^2}{v} e^{v(y + \eta)} - \frac{\pi}{2} v e^{v(y + \eta)} \right] \\
 &\quad + 2v e^{vy} \int_{-\infty}^0 f(\eta) e^{v\eta} d\eta
 \end{aligned}$$

$$= \frac{1}{\pi} \int_0^{\infty} dk \int_{-\infty}^{\infty} f(\eta) d\eta \cos k(y - \eta), \text{ with } f(-\eta) = f(\eta)$$

$$= f(y)$$

by the usual Fourier integral theorem. We have made use above of the following two integrals:

$$\int_0^{\infty} \frac{\cos mx}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-|ma|}; \int_0^{\infty} \frac{x \sin mx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ma}, m > 0, a > 0$$

Further separation of variables should now depend upon the nature of the problem to be solved. In one that we shall consider below, a wavemaker at one end of a rectangular channel, it is natural to use again rectangular coordinates and to assume $\chi(x, z) = X(x)Z(z)$. If one wished to find the wave motion generated by an oscillating vertical post of circular cross section, one would use polar coordinates. However, as mentioned earlier, the number of possible configurations is very limited.

We shall determine the functions $X(x)$ and $Z(z)$ for separation in rectangular coordinates. Substitution into (2.11) quickly gives

$$\frac{X''}{X} + \frac{Z''}{Z} + k^2 = 0 \quad (2.24)$$

or

$$X'' - \alpha X = 0, Z'' - \gamma Z = 0, \alpha + \gamma = -k^2 < 0$$

If $\gamma = q^2 > 0$, then

$$Z = C e^{qz} + D e^{-qz}$$

and

$$X = E \cos (k^2 + q^2)^{1/2} x + F \sin (k^2 + q^2)^{1/2} x \quad (2.25)$$

If $\gamma = -q^2 < 0$, then

$$Z = C \cos qz + D \sin qz \quad (2.26)$$

and

$$X = E \cos (k^2 - q^2)^{1/2} x + F \sin (k^2 - q^2)^{1/2} x \quad \text{if } q^2 < k^2$$

$$X = E \exp (q^2 - k^2)^{1/2} x + F \exp [-(q^2 - k^2)^{1/2} x] \quad \text{if } q^2 > k^2$$

If we substitute into (2.15), we again obtain (2.24) except that now $\alpha + \gamma = k^2$. Now if $\gamma = q^2 > 0$,

$$Z = C e^{qz} + D e^{-qz} \quad (2.27)$$

and

$$X = E \cos (q^2 - k^2)^{1/2} x + F \sin (q^2 - k^2)^{1/2} x \quad \text{if } q^2 > k^2$$

$$X = E \exp (k^2 - q^2)^{1/2} x + F \exp [-(k^2 - q^2)^{1/2} x] \quad \text{if } q^2 < k^2$$

If $\gamma = -q^2 < 0$, then

$$Z = C \cos qz + D \sin qz \quad (2.28)$$

and

$$X = E \exp (k^2 + q^2)^{1/2} x + F \exp [-(k^2 + q^2)^{1/2} x]$$

This is a complete census of the possibilities in rectangular coordinates.

Let us now turn to some applications. The first one is a wavemaker at one end of a semi-infinite rectangular channel. The second is a wavemaker in a wall bounding the region $x > 0$, $y < 0$. The treatments parallel

each other except that the y - and z -dimensions are bounded in one, and infinite in the other. The first employs Fourier series in the solution, the second Fourier integrals.

WAVEMAKER IN A CHANNEL

Consider a semi-infinite channel bounded by plane walls at $y = -h$ and $z = 0$ and $z = b$. At $x = 0$ there is situated a wavemaker that moves according to the equation

$$x = F(y, z) \sin \sigma t \quad (2.29)$$

The channel extends to infinity in the direction Ox . In practice, this can be approximated by having a very efficient wave absorber at the end away from the wavemaker. A wide variety of wavemakers, both two- and three-dimensional, can be described by a proper choice of $F(y, z)$. We shall further assume that the motion has persisted for a long time, so that transient motions associated with starting the wavemaker have died out and the fluid motion is also harmonic with frequency σ .

Let us formulate the mathematical problem. We may evidently take ϕ in the form (1.12) and the free-surface condition in the form (1.13). The boundary conditions on the sides and bottom will be

$$\phi_z(x, y, 0) = \phi_z(x, y, b) = 0, \quad \phi_y(x, -h, z) = 0 \quad (2.30)$$

The linearized boundary condition on the wavemaker corresponding to (1.6) is

$$\phi_x(0, y, z, t) = \sigma F(y, z) \cos \sigma t$$

or

$$\phi_{1x}(0, y, z) = \sigma F, \quad \phi_{2x}(0, y, z) = 0 \quad (2.31)$$

There is still one missing condition, but it will be more interesting to see this forced upon us later on.

In order to solve this problem, we shall try to use the family of solutions given in (2.17) together with the solutions for χ given in (2.25)-(2.28). First of all we note that the boundary conditions on the side walls given in (2.30) cannot both be satisfied with any combination of the exponential solutions (2.25) or (2.27). These conditions will be satisfied by the trigonometric solutions (2.26) and (2.28) if we take C and D such that

$$Z = \cos \frac{m\pi}{b} z, \quad m = 0, 1, 2, \dots \quad (2.32)$$

In choosing the X's to go with the Z's, we must remember that in (2.26) k must be k_0 and in (2.28) k may be any one of the k_i (see (2.17)). Finally, since we are interested only in bounded solutions, we must discard the increasing exponential in the exponential solutions for X. If we now sum all these elementary solutions with arbitrary multipliers, we anticipate that the solution to the problem can be written in the following form:

$$\begin{aligned} \phi(x, y, z) = & \sum_{m=0}^M \left\{ A_m \cos \left[k_0^2 - \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} x - b_m \sin \left[k_0^2 - \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} x \right\} \\ & \cosh k_0(y + h) \cos \frac{m\pi}{b} z \\ & + \sum_{m=M+1}^{\infty} b_m \exp \left[- \left(\left(\frac{m\pi}{b} \right)^2 - k_0^2 \right)^{1/2} x \right] \cosh k_0(y + h) \cos \frac{m\pi}{b} z \\ & + \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} b_{im} \exp \left[- \left(k_i^2 + \left(\frac{m\pi}{b} \right)^2 \right)^{1/2} x \right] \cos k_i(y + h) \cos \frac{m\pi}{b} z \quad (2.33) \end{aligned}$$

where M is the largest integer m such that $\frac{m\pi}{b} < k_0$.

This function satisfies the Laplace equation, the free-surface condition, and the conditions on the bottom and sides of the channel; ϕ may be either ϕ_1 or ϕ_2 .

Let us examine the condition (2.31) on the wavemaker:

$$\begin{aligned}\phi_x(0, y, z) = & \sum_{m=0}^M -b_m \left[k_0^2 - \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} \cosh k_0(y+h) \cos \frac{m\pi}{b} z \\ & + \sum_{m=M+1}^{\infty} -b_m \left[\left(\frac{m\pi}{b} \right)^2 - k_0^2 \right]^{1/2} \cosh k_0(y+h) \cos \frac{m\pi}{b} z \\ & + \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} -b_{im} \left[k_i^2 + \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} \cos k_i(y+h) \cos \frac{m\pi}{b} z \quad (2.34)\end{aligned}$$

If we now use the orthogonality of the functions $\{\cosh k_0(y+h), \cos k_i(y+h), i = 1, 2, \dots\}$ Stated in (2.19) and the well-known orthogonality of the family $\{\cos \frac{m\pi}{b} z, m = 0, 1, 2, \dots\}$ on the interval $0 \leq z \leq b$, we may easily derive the following formulas for the b 's:

$$\begin{aligned}& -\frac{b}{8k_0} (\sinh 2k_0 h + 2k_0 h) \left[k_0^2 - \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} b_m = \\ & = \int_{-h}^0 dy \int_0^b dz \phi_x(0, y, z) \cosh k_0(y+h) \cos \frac{m\pi}{b} z, \quad m < M \\ & -\frac{b}{8k_0} (\sinh 2k_0 h + 2k_0 h) \left[\left(\frac{m\pi}{b} \right)^2 - k_0^2 \right]^{1/2} b_m \\ & = \int_{-h}^0 dy \int_0^b dz \phi_x(0, y, z) \cosh k_0(y+h) \cos \frac{m\pi}{b} z, \quad m \geq M+1\end{aligned} \quad (2.35)$$

$$-\frac{b}{8k_1} (\sin 2k_1 h + 2k_1 h) \left[k_1^2 + \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} b_m$$

$$= \int_{-h}^0 dy \int_0^b dz \phi_x(0, y, z) \cos k_1 (y + h) \cos \frac{m\pi}{b} z, \quad i \geq 1, \quad m \geq 0$$

In order to determine ϕ_1 , we replace ϕ_x in (2.35) by σF ; to determine ϕ_2 , by 0. As long as k_0 is such that it is never equal to $m\pi/b$ for some integer m , all coefficients b are uniquely determined. The completeness of the two orthogonal families for representing functions on $-h \leq y \leq 0$ and $0 \leq z \leq b$ tells us that any square-integrable function F can be so represented. For ϕ_2 , the b 's are obviously all zero.

Have we now determined the functions ϕ_1 and ϕ_2 ? Evidently not, for a_0, \dots, a_M are not determined for either ϕ_1 or ϕ_2 . Since we have now satisfied all our boundary conditions, we must conclude that either the problem does not have a unique solution or else that we have not completely formulated it. Here the latter is true. If there had been another wall at $x = \ell > 0$, we should have had another boundary condition to determine the coefficients. We evidently need something to replace it. This is the radiation condition, which states that waves must propagate down the channel, i.e., in the direction Ox . The terms exponential in x play no role in this condition for they die out as x increases, representing only a local disturbance near the wavemaker. In order to apply the radiation condition, we may write the solution in the complete form

$$\Phi(x, y, z, t) = \phi_1 \cos \sigma t + \phi_2 \sin \sigma t$$

and then choose the a 's so that each of these first M terms represents a progressive wave moving to the right. It is easy to see that this is achieved by taking $a_m^{(1)} = 0$ and $a_m^{(2)} = b_m^{(1)}$, where the superscripts

respectively refer to ϕ_1 and ϕ_2 , for then the first M terms take the form

$$\sum_{m=0}^M -b_m^{(1)} \sin \left\{ \left[k_0^2 - \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} x - \sigma t \right\} \cosh k_0(y+h) \cos \frac{m\pi}{b} z$$

This choice of the a 's has been forced upon us by the radiation condition, the latter being necessary to achieve a unique solution. However, if the problem had been formulated as an initial-value problem in which the motion started from rest, this solution would have been obtained automatically.

It will be convenient for discussion of its properties to write the solution in a slightly different form. Define

$$C_m = \frac{8k_0}{b [\sinh 2k_0 h + 2k_0 h]} \int_{-h}^0 dy \int_0^b dz F(y,z) \cosh k_0(y+h) \cos \frac{m\pi}{b} z \quad (2.36)$$

$$C_{im} = \frac{8k_i}{b [\sin 2k_i h + 2k_i h]} \int_{-h}^0 dy \int_0^b dz F(y,z) \cos k_i(y+h) \cos \frac{m\pi}{b} z$$

Then Φ is given by

$$\begin{aligned} \Phi(x,y,z,t) = & \sum_{m=0}^M \frac{\sigma}{\left[k_0^2 - \left(\frac{m\pi}{b} \right)^2 \right]^{1/2}} C_m \sin \left\{ \left[k_0^2 - \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} x - \sigma t \right\} \\ & \times \cosh k_0(y+h) \cos \frac{m\pi}{b} z \end{aligned}$$

$$\begin{aligned}
& - \sum_{m=M+1}^{\infty} \frac{\sigma}{\left[\left(\frac{m\pi}{b} \right)^2 - k_0^2 \right]^{1/2}} C_m \exp \left\{ - \left[\left(\frac{m\pi}{b} \right)^2 - k_0^2 \right]^{1/2} x \right\} \cosh k_0 (y + h) \\
& \quad \times \cos \frac{m\pi}{b} z \cos \sigma t \\
& - \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} \frac{\sigma}{\left[k_i^2 + \left(\frac{m\pi}{b} \right)^2 \right]^{1/2}} C_{im} \exp \left\{ - \left[k_i^2 + \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} x \right\} \cos k_i (y + h) \\
& \quad \times \cos \frac{m\pi}{b} z \cos \sigma t
\end{aligned} \tag{2.37}$$

According to (1.10), the free surface itself is given by $Y = -g^{-1} \phi_t(x, 0, z, t)$. It is as follows:

$$\begin{aligned}
Y(x, z, t) &= \sum_{m=0}^M \frac{k_0 \sinh k_0 h}{\left[k_0^2 - \left(\frac{m\pi}{b} \right)^2 \right]^{1/2}} C_m \cos \left\{ \left[k_0^2 - \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} x - \sigma t \right\} \cos \frac{m\pi}{b} z \\
&- \sum_{m=M+1}^{\infty} \frac{k_0 \sinh k_0 h}{\left[\frac{m\pi^2}{b} - k_0^2 \right]^{1/2}} C_m \exp \left\{ - \left[\left(\frac{m\pi}{b} \right)^2 - k_0^2 \right]^{1/2} x \right\} \cos \frac{m\pi}{b} z \sin \sigma t \\
&+ \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} \frac{k_i \sin k_i h}{\left[k_i^2 + \left(\frac{m\pi}{b} \right)^2 \right]^{1/2}} C_{im} \exp \left\{ - \left[k_i^2 + \left(\frac{m\pi}{b} \right)^2 \right]^{1/2} x \right\} \cos \frac{m\pi}{b} z \sin \sigma t
\end{aligned} \tag{2.38}$$

Let us examine the solution. It is evident that it breaks down for any k_0 such that $k_0 = m\pi/b$ for some integer m . Indeed, the formulas determining the b_m in (2.35) do not do so for the particular m for which this happens, so that we have not, in fact, found a solution for such a value

of k_0 (or its associated frequency σ). Hence, there is an infinite sequence $\sigma_1, \sigma_2, \sigma_3, \dots$ for which no solution appears to exist. Let us examine the behavior of (2.38) in the neighborhood of these frequencies.

We shall make a thought experiment in which the frequency σ starts out quite small and then increases little by little through these special values σ_1 (Figure 3).

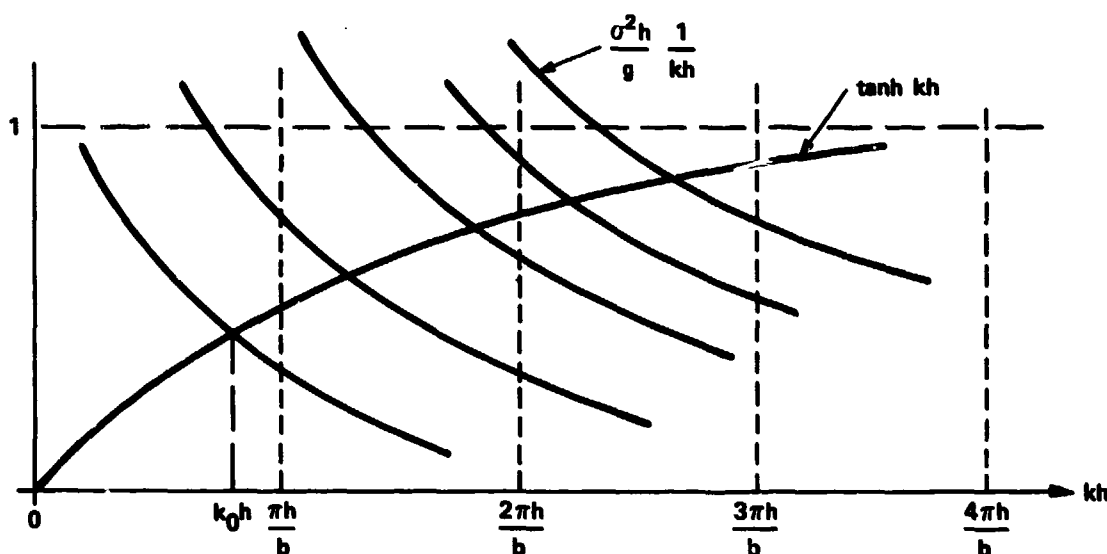


Figure 3 - Critical Frequencies for Wavemaker in a Channel

When σ is quite small, $k_0 h < \pi h/b$ and $M = 0$. After the local disturbance has died out, there is then a single two-dimensional progressive wave propagating down the channel:

$$C_0 \sinh k_0 h \cos (k_0 x - \sigma t)$$

However, as σ approaches σ_1 from below, $k_0 h \rightarrow \pi h/b$ and the coefficient of the first exponential term in the second summation

$$C_1 \frac{k_0 \sin k_0 h}{\left[\left(\frac{\pi}{b}\right)^2 - k_0^2\right]^{1/2}} \exp \left\{ - \left[\left(\frac{\pi}{b}\right)^2 - k_0^2 \right]^{1/2} x \right\} \cos \frac{\pi}{b} z \sin \sigma t$$

grows unboundedly. At the same time the coefficient of x in the exponential decreases, so that the disturbance no longer dies out so quickly.

If σ is increased just beyond σ_1 , then $M = 1$ and the first summation consists of two terms

$$C_0 \sinh k_0 h \cos (k_0 x - \sigma t)$$

and

$$C_1 \frac{k_0 \sinh k_0 h}{\left[k_0^2 - \left(\frac{\pi}{b}\right)^2\right]^{1/2}} \cos \left\{ \left[k_0^2 - \left(\frac{\pi}{b}\right)^2 \right]^{1/2} x - \sigma t \right\} \cos \frac{\pi}{b} z$$

The first is, of course, a two-dimensional progressive wave. The second represents a progressive wave of longer length sloshing from side to side as it propagates. It might be indicated schematically as shown in Figure 4. If σ is close to σ_1 , its

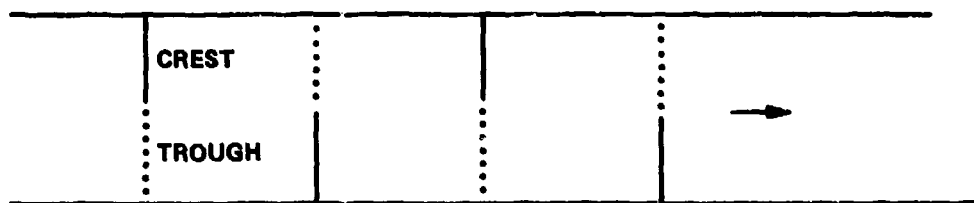


Figure 4 - Representation of Sloshing Progressive Wave

amplitude will be very large. As σ increases, its amplitude will decrease and approach $C_1 \sinh k_0 h$. However, before σ increases very much, it will approach σ_2 and another "catastrophe" will occur. After passing σ_2 , a third type of progressive wave modulated by $\cos (2\pi/b)z$ will be added to the first two (see Figure 5).

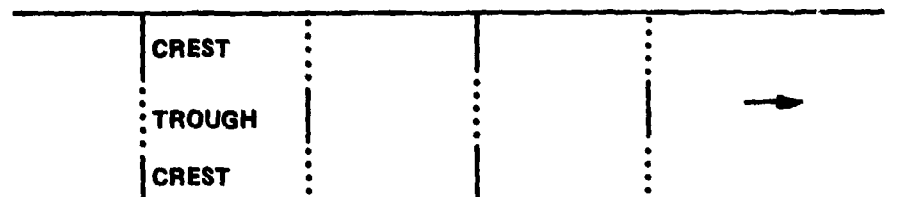


Figure 5 - Representation of Third Type of Progressive Wave

It is evident that every time σ passes through a value σ_m , there is an associated crisis during which a new progressive wave is added, beginning with a very large amplitude. These frequencies are sometimes called "cutoff frequencies." One should note that even though the wavemaker is carrying out a three-dimensional motion, the resulting progressive wave will be two-dimensional if σ is small enough, i.e., $\sigma < \sigma_1$. On the other hand, if the motion of the wavemaker is exactly two-dimensional, so that $F(y, z) = F(y)$, then $C_m = 0$ for $m \geq 1$ and also $\sigma_{im} = 0$ for $m \geq 1$. Thus, none of these crises occurs. However, if the supposed two-dimensional wavemaker is only a little out of true, one may anticipate the catastrophic behavior described above.

One may argue legitimately, that the linearized theory is no longer a valid approximation in the neighborhood of these cutoff frequencies. However, the behavior described above does occur and is well known. It is not the same as a similar phenomenon known as "cross-waves," recently studied by Garrett (1970), Mahony (1972), and others (Garrett gives a history of this phenomenon). Cross-waves are a nonlinear phenomenon. I am not sure who first solved the problem we have just examined. It is included in the Biesel and Suquet's (1951) encyclopedic article on wavemakers, and certain aspects have been studied in more detail by Kravtchenko (1954). However, it is accessible to anyone familiar with the methods of classical mathematical physics.

One question may remain. Since we have not found a solution for the cutoff frequencies σ_1 , what should be done about them? These are like the resonance frequencies in forced motion of a harmonic oscillator. One must reformulate the problem as an initial-value problem. Then one may anticipate that the solution for these frequencies will grow without bound as $t \rightarrow \infty$.

WAVEMAKER IN A WALL

As an example of the use of Fourier integrals and of the representation theorem (2.22), (2.23), we have chosen a problem similar to the one just discussed. The chief difference is that the bottom and side walls are removed so that the only boundary is the wall at $x = 0$. The fluid is in the region $x > 0$, $y < 0$. The wall itself is flexible and moves according to the equation

$$x = F(y, z) \sin \sigma t \quad (2.39)$$

where we assume that

$$\int_{-\infty}^0 dy \int_{-\infty}^{\infty} dz |F(y, z)| < \infty \quad (2.40)$$

As before, we assume that the motion has become harmonic in time.

The mathematical formulation is similar to that of the wavemaker in a channel. We take ϕ in the form (1.12) and the free-surface condition in the form (1.13). The first two conditions of (2.30) are abandoned and the third replaced by

$$\lim_{y \rightarrow -\infty} \phi_y(x, y, z) = 0 \quad (2.41)$$

The boundary condition (2.31) still holds. In addition, here, as before, we need an explicit condition stating that the waves propagate away from the wavemaker.

We must now make a selection of the solutions (2.21), (2.25)-(2.28). The exponential solutions in z must be discarded because they become unbounded on one side or the other. The decreasing exponentials in x are allowable. The most general solution for ϕ that satisfies these conditions is the following:

$$\begin{aligned} \phi(x,y,z) = & \int_0^v dq \left\{ \cos [(\nu^2 - q^2)^{1/2} x] [A_0(q) \cos qz + B_0(q) \sin qz] \right. \\ & \left. - \sin [(\nu^2 - q^2)^{1/2} x] [C_0(q) \cos qz + D_0(q) \sin qz] \right\} e^{\nu y} \\ & + \int_v^\infty dq \exp [-(q^2 - \nu^2)^{1/2} x] [C_0(q) \cos qz + D_0(q) \sin qz] e^{\nu y} \\ & + \int_0^\infty dk \int_0^\infty dq \exp [-(q^2 + \nu^2)^{1/2} x] [C(k,q) \cos qz + D(k,q) \sin qz] \\ & \times [k \cos kz + \nu \sin ky] \end{aligned} \quad (2.42)$$

Here, ϕ can be either ϕ_1 or ϕ_2 . We can confirm immediately that (2.41) is satisfied if the integrals exist. For the single integrals, this follows from the form of the factor $e^{\nu y}$. For the double integral, this follows from the Riemann-Lebesgue Lemma.

Let us now compute

$$\begin{aligned}
\phi_x(0,y,z) = & - \int_0^v dq (v^2 - q^2)^{1/2} [C_0(q) \cos qz + D_0(q) \sin qz] e^{vy} \\
& - \int_v^\infty dq (q^2 - v^2)^{1/2} [C_0(q) \cos qz + D_0(q) \sin qz] e^{vy} \\
& - \int_0^\infty dk \int_0^\infty dq (q^2 + k^2)^{1/2} [C(k,q) \cos qz + D(k,q) \sin qz] \\
& \times [k \cos ky + v \sin ky]
\end{aligned} \tag{2.43}$$

We should like to invert this to obtain C_0, D_0, C , and D in terms of ϕ_x . To do this we shall need both the ordinary Fourier transform as well as (2.23). We recall that the former may be written as follows: If

$$f(z) = \int_0^\infty dq [A(q) \cos qz + b(q) \sin qz] = \operatorname{Re} \int_0^\infty dq (a + ib) e^{-iqz}$$

then

$$a + ib = \frac{1}{\pi} \int_{-\infty}^\infty dz f(z) e^{iqz}$$

We apply (2.23) first and treat z as a fixed parameter. Then

$$\begin{aligned}
b_0(z) &= - \int_0^{\infty} dq |v^2 - q^2|^{1/2} [C_0(q) \cos qz + D_0(q) \sin qz] \\
&= 2v \int_{-\infty}^0 \phi_x(0, y, z) e^{vy} dy
\end{aligned}$$

and

$$\begin{aligned}
b(k, z) &= - \int_0^{\infty} dq (q^2 + k^2)^{1/2} [C(k, q) \cos qz + D(k, q) \sin qz] \\
&= \frac{2}{\pi} \int_{-\infty}^0 \phi_x(0, y, z) \frac{k \cos ky + v \sin ky}{k^2 + v^2} dy
\end{aligned}$$

The ordinary Fourier inversion integral now gives

$$- |v^2 - q^2|^{1/2} [C_0(q) + iD_0(q)] = \frac{2v}{\pi} \int_{-\infty}^0 dy \int_{-\infty}^{\infty} dz \phi_x(0, y, z) e^{vy} e^{iqz} \quad (2.45)$$

and

$$\begin{aligned}
- (k^2 + q^2)^{1/2} [C(k, q) + iD(k, q)] &= \frac{2}{\pi^2} \int_{-\infty}^0 dy \int_{-\infty}^{\infty} dz \phi_x(0, y, z) \\
&\quad \frac{k \cos ky + v \sin ky}{k^2 + v^2} e^{iqz}
\end{aligned}$$

The coefficients $A_0(q)$ and $B_0(q)$ have not yet been determined for either ϕ_1 or ϕ_2 . For ϕ_2 , since $\phi_{2x}(0, y, z) = 0$, all the C's and D's are zero and ϕ_2 takes the form

$$\phi_2(x, y, z) = \int_0^v dq \cos[(v^2 - q^2)^{1/2} x] [A_{02} \cos qz + B_{02} \sin qz] e^{vy}$$

If we consider only the part of $\phi_1 \cos \sigma t + \phi_2 \sin \sigma t$ involving the first integral, we find

$$\begin{aligned} \int_0^v dq \left\{ \cos[(v^2 - q^2)^{1/2} x] [A_{01} \cos qz + B_{01} \sin qz] \cos \sigma t \right. \\ \left. - \sin[(v^2 - q^2)^{1/2} x] [C_{01} \cos qz + D_{01} \sin qz] \cos \sigma t \right. \\ \left. + \cos[(v^2 - q^2)^{1/2} x] [A_{02} \cos qz + B_{02} \sin qz] \sin \sigma t \right\} e^{vy} \end{aligned}$$

In order that this should represent an outgoing wave, we must set

$$A_{01} = B_{01} \equiv 0, A_{02} = +C_{01}, B_{02} = +D_{01}$$

This gives for this integral

$$\begin{aligned} & - \int_0^v dq \sin[(v^2 - q^2)^{1/2} x - \sigma t] [C_{01} \cos qz + D_{01} \sin qz] e^{vy} \\ & = \frac{-1}{2} \int_0^v dq \left\{ C_{01} \sin[(v^2 - q^2)^{1/2} x + qz - \sigma t] + C_{01} \sin[(v^2 - q^2)^{1/2} x \right. \\ & \quad \left. - qz - \sigma t] - D_{01} \cos[(v^2 - q^2)^{1/2} x + qz - \sigma t] + D_{01} \sin[(v^2 - q^2)^{1/2} x \right. \\ & \quad \left. - qz - \sigma t] \right\} e^{vy} \end{aligned} \quad (2.46)$$

The functions C_{01} , D_{01} , C , and D are, of course, known from (2.45) with ϕ_x replaced by $\sigma F(y, z)$.

The solution has no especially interesting properties like those occurring in the channel. This fact is associated not with the infinite depth but with the infinite width. The cutoff frequencies would also have occurred here if there had been walls at $z = 0$ and $z = b$. In this case the solution would have been a bit different in that we would have used Fourier series in the z -direction rather than the Fourier integral.

Let us compute the average rate at which work is being done by the wavemaker. According to the linear approximation, the instantaneous rate is

$$\begin{aligned} W(t) &= \int_{-\infty}^0 dy \int_{-\infty}^{\infty} dz \, p(0, y, z) \, u(0, y, z) = -\rho \int_{-\infty}^0 dy \int_{-\infty}^{\infty} dz \, \phi_x(0, y, z, t) \, \phi_t(0, y, z, t) \\ &= -\rho \iint dy dz \, [\phi_{1x} \cos \sigma t + \phi_{2x} \sin \sigma t] \, \sigma \, [-\phi_1 \sin \sigma t + \phi_2 \cos \sigma t] \\ &= -\rho \sigma \iint dy dz \, [\phi_{1x} \phi_2 \cos^2 \sigma t - \phi_{2x} \phi_1 \sin^2 \sigma t \\ &\quad + (-\phi_{1x} \phi_1 + \phi_{2x} \phi_2) \sin \sigma t \cos \sigma t] \\ &= -\rho \sigma^2 \iint dy dz \, F(y, z) \, [\phi_2 \cos^2 \sigma t - \phi_1 \sin \sigma t \cos \sigma t] \end{aligned}$$

where we have made use of the boundary conditions for ϕ_{1x} and ϕ_{2x} when $x = 0$. If we wish to find only the average rate, we must still integrate with respect to t over one period $2\pi/\sigma$. This yields

$$\begin{aligned} \bar{W} &= \frac{-1}{2} \rho \sigma^2 \iint dy dz \, F(y, z) \, \phi_2(0, y, z) \\ &= + \frac{1}{2} \rho \sigma^2 \iint dy dz \, F(y, z) \int_0^v dq \, [C_{01} \cos qz + D_{01} \sin qz] e^{vy} \\ &= + \frac{1}{2} \rho \sigma^2 \iint dy dz \, F(y, z) \int_0^v dq \, [C_{01} + iD_{01}] e^{-iqz} e^{vy} \end{aligned}$$

Using (2.45) with $\phi_{1x}(0, y, z) = \sigma F(y, z)$, we find

$$\begin{aligned}
 \bar{W} &= \frac{-1}{2} \rho \sigma^3 \iint dy dz F(y, z) \int_0^v dq \frac{e^{vy} e^{-iqz}}{(v^2 - q^2)^{1/2}} \frac{2v}{\pi} \iint d\eta d\zeta F(\eta, \zeta) e^{v\eta} e^{iq\zeta} \\
 &= \frac{-1}{\pi} \rho \frac{\sigma^5}{g} \int_{-\infty}^0 dy \int_{-\infty}^{\infty} dz \int_{-\infty}^0 d\eta \int_{-\infty}^{\infty} d\zeta F(y, z) F(\eta, \zeta) \int_0^v dq \frac{\cos q(z - \zeta)}{(v^2 - q^2)^{1/2}} e^{v(y+\eta)} \\
 &= \frac{-1}{2} \rho \frac{\sigma^5}{g} \int_{-\infty}^0 dy \int_{-\infty}^{\infty} dz \int_{-\infty}^0 d\eta \int_{-\infty}^{\infty} d\zeta F(y, z) F(\eta, \zeta) e^{v(y+\eta)} J_0 [v(z - \zeta)]
 \end{aligned}
 \tag{2.47}$$

This formula is somewhat analogous to the Michell integral for the resistance of a thin ship. Indeed, the latter problem is a kind of steady-state analog in which one considers the flow past a small (\approx "thin") bump in a wall. In fact, Michell derived this wave-resistance formula by a Fourier-integral method very similar to that just used; it is described in Wehausen (1973, pp. 143-148).

The problem discussed above was first solved by Havelock (1929). As he pointed out, (2.22) or (2.23) can be used together with any other coordinate system in which one can separate the function $\chi(x, z)$ of (2.21). He applies the method to the waves generated by the oscillation of a vertical circular post, but he considers only the simplest case when there is no dependence upon the angles. An oscillation with angular dependence would lead to situations of the sort encountered with the wavemaker in a channel.

The problem we have considered is essentially a simple one. However, the method can be and has been applied to more complicated situations. For example, Ursell (1947) has used it to investigate diffraction of water waves from an obstacle in the form of a vertical plate of finite length, treating it as a two-dimensional problem with normal incidence on the plate. Evans

and Morris (1972) have extended the treatment to oblique incidence. Such problems require finding a solution on each side of the plate and then matching them at the interface below the plate. In addition, one must take into account conditions at the edge of the plate. A method that can be used to solve a diffraction problem can also be used to solve a related forced-motion problem.

PART 3: THE REDUCTION METHOD

The essence of this method is to consider at first the function $\phi_{tt} + g\phi_y$ instead of ϕ itself. This function is also a solution of the Laplace equation and vanishes on the surface $y = 0$. Let us define

$$H(x, y, z, t) = \phi_{tt} + g\phi_y \quad (3.1)$$

We extend the region of definition of H from those parts of the lower half-space $y < 0$ occupied by fluid to their mirror images in $y > 0$ by the equation

$$H(x, -y, z, t) = -H(x, y, z, t) \quad (3.2)$$

Since $H(x, -0, z, t) = 0$, it follows immediately that not only is H continuous on $y = 0$ but also H_x , H_z , H_{xx} , and H_{zz} . However, it also follows from the definition that $H_y(x, -0, z, t) = H_y(x, +0, z, t)$ so that H_y is also continuous. Furthermore, since $H_{yy} = -H_{xx} - H_{zz}$, it follows that also H_{yy} is continuous. Hence, H has been extended as a harmonic function into the upper half-space. If the rest of the boundary conditions can be carried over to H and do not complicate the problem, the solution for H may turn out to be easier than that for ϕ . Of course, after finding H , we are still confronted with the problem of finding ϕ , which may not be easy.

Although the method is presented above in a three-dimensional formulation, it has found its greatest use in two-dimensional problems where the powerful methods of analytic-function theory may be applied. We make a small digression to review some of the basic facts concerning irrotational, two-dimensional flow of an incompressible fluid.

The equations expressing conservation of mass and irrotationality

$$u_x + v_y = 0, \quad u_y - v_x = 0 \quad (3.3)$$

are just the Cauchy-Riemann equations for u and $-v$, and consequently

$w = u - iV$ is an analytic function of $z = x + iy$. Furthermore, the first equation in (3.3) implies that there exists a stream function $\Psi(x, y, t)$ with $\Psi_x = -V$, $\Psi_y = u$. But then ϕ and Ψ also satisfy the Cauchy-Riemann equations and

$$F(z, t) = \phi(x, y, t) + i\Psi(x, y, t) \quad (3.4)$$

is also an analytic function. It is easy to see that

$$F'(z, t) = w \quad (3.5)$$

We shall call F the complex potential and w the complex velocity.

Let us now consider the combination

$$\begin{aligned} H(z) &= F_{tt} + igF' = \phi_{tt} + i\Psi_{tt} + ig(\phi_x - i\phi_y) \\ &= \phi_{tt} + g\phi_y + i(\Psi_{tt} + g\Psi_y) \end{aligned} \quad (3.6)$$

It follows from the free-surface condition that

$$\operatorname{Re} H(x + i0) = 0 \quad (3.7)$$

But then $H(z)$ can be extended as an analytic function from the domain $y \leq 0$ to the domain $y \geq 0$ by the Schwarz reflection principle:

$$H(x - iy) = -\overline{H(x + iy)} \quad (3.8)$$

This of course, is the analogue of (3.2) and as in the three-dimensional case, we hope to be able to exploit this extension to the whole plane to find an easier solution for H .

The method has been used by so many persons that I shall not try to sketch its history except to say that I believe it was first used in water-wave problems by Levi-Civita. It has been extensively exploited by various Russian hydrodynamicists but also by others in the United States and elsewhere. To illustrate the method, we shall use diffraction from a vertical plate immersed into the water to a depth ℓ . This problem was treated in this way by Haskind (1948). It can be treated equally well by the method of Fourier integrals, and in fact this was done by Ursell (1947). The method has recently been applied by Evans (1970) to the problem of diffraction about a completely submerged flat plate. It is almost obvious that if the diffraction problem can be solved by this method, then the problem of the waves generated by small oscillations of the plate can also be solved. This is also included in the cited paper by Evans. Later on we shall come back to the reduction method when we treat the method of multipole expansions.

DIFFRACTION OF WAVES ON A VERTICAL PLATE

We suppose that a flat plate is immersed to a depth ℓ in an infinitely deep fluid and is subjected to oncoming waves. Some part of these will be reflected and some part transmitted, as indicated schematically in Figure 6. In the neighborhood of the plate there will be a local disturbance that we have not tried to represent.

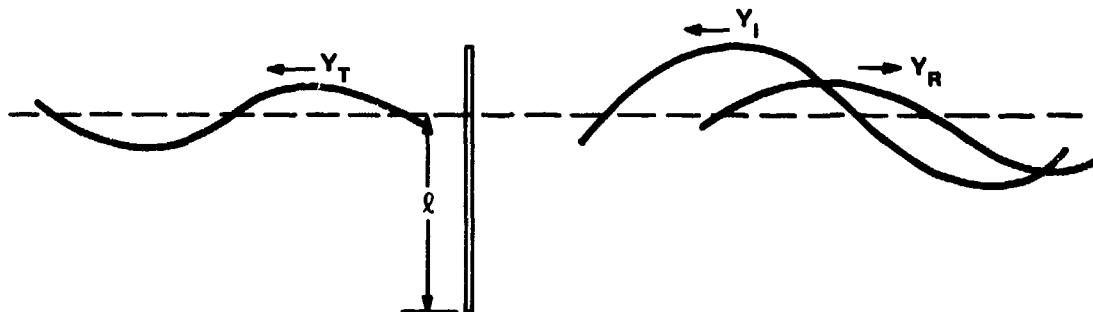


Figure 6 - Flat Plate in Oncoming Waves

The incident wave will be represented by

$$Y_I = A \cos (kx + \sigma t), \quad \sigma^2 = gk \quad (3.9)$$

with velocity potential

$$\begin{aligned} \phi_I &= -\frac{gA}{\sigma} e^{ky} \sin (kx + \sigma t) \\ &= -\frac{gA}{\sigma} e^{ky} \sin kx \cos \sigma t - \frac{gA}{\sigma} e^{ky} \cos kx \sin \sigma t \\ &= \phi_{IC} \cos \sigma t + \phi_{IS} \sin \sigma t \end{aligned} \quad (3.10)$$

The associated complex potential of (3.4) is easily verified to be

$$\begin{aligned} F_I(z) &= -i \frac{gA}{\sigma} e^{-ikz} \cos \sigma t - \frac{gA}{\sigma} e^{-ikz} \sin \sigma t \\ &= f_{IC} \cos \sigma t + f_{IS} \sin \sigma t \end{aligned} \quad (3.11)$$

We shall denote the diffracted wave by ϕ_D so that the velocity potential for the total motion is

$$\begin{aligned} \phi &= \phi_I + \phi_D = \phi_C \cos \sigma t + \phi_S \sin \sigma t \\ &= (\phi_{IC} + \phi_{DC}) \cos \sigma t + (\phi_{IS} + \phi_{DS}) \sin \sigma t \end{aligned} \quad (3.12)$$

Analogously, the complex potential is

$$\begin{aligned} F &= F_I + F_D = f_C \cos \sigma t + f_S \sin \sigma t \\ &= (f_{IC} + f_{DC}) \cos \sigma t + (f_{IS} + f_{DS}) \sin \sigma t \end{aligned} \quad (3.13)$$

The following boundary conditions must be satisfied on the flat plate:

$$\phi_x(0, y, t) = 0, \text{ or } \phi_{CX}(0, y) = \phi_{SX}(0, y) = 0, \quad 0 \leq y < -l \quad (3.14)$$

On the free surface, (1.11) must be satisfied by ϕ and ϕ_D and hence (1.13) by ϕ_C , ϕ_S , ϕ_{DC} , and ϕ_{DS} . In complex form these are the following:

On the plate:

$$\operatorname{Re} f'_C(0 + iy) = \operatorname{Re} f'_S(0 + iy) = 0, \quad 0 \leq y < -l \quad (3.15)$$

or in terms of F_D ,

$$\operatorname{Re} \left\{ i\sigma e^{ky} + f'_{DC}(0 + iy) \right\} = \operatorname{Re} f'_{DS}(0 + iy) = 0 \quad (3.16)$$

On the free surface:

$$\operatorname{Im} \{ f' + ikf(x + i0) \} = 0 \quad (3.17)$$

where f is any one of f_C , f_S , f_{DC} , f_{DS} .

In addition, ϕ_D must represent outgoing waves at a distance from the plate. In complex notation this may be expressed as follows:

$$\lim_{x \rightarrow +\infty} (f'_{DC} - kf_{DS}) = 0, \quad \lim_{x \rightarrow +\infty} (f'_{DS} + kf_{DC}) = 0 \quad (3.18)$$

It is also necessary to specify the behavior of the flow near the sharp edge at $(0, -l)$ for there will be a singularity at this point. In order to limit the power of this singularity, we shall suppose that

$$|z + i\ell|^{1/2} |f'(z)| < M \quad \text{near } z = -i\ell$$

or in other words that

$$(z + i\ell)^{1/2} f'(z) \quad (3.19)$$

is regular at the point $-i\ell$. This restriction is in conformity with experience at similar cusps in hydrodynamics.

Finally we impose some further conditions at infinity:

$$|f'(z)| < B \text{ if } |z| \geq \ell + 1, y < 0 \quad (3.20)$$

and

$$\lim_{y \rightarrow -\infty} f'(z) = 0 \quad (3.21)$$

Until we reach the point where we must deal with the radiation condition, we may safely omit the indices c and s for the calculations are the same for each.

Let us now introduce the auxiliary function

$$G(z) = f'(z) + ikf(z) \quad (3.22)$$

which according to (3.17) satisfies the condition

$$\text{Im } G(x + i0) = 0 \quad (3.23)$$

By the Schwarz reflection principle, we extend G analytically into the upper half-plane:

$$G(x + iy) = \overline{G(x - iy)} \quad (3.24)$$

or

$$\phi_x(x, y) - k\psi(x, y) = \phi_x(x, -y) - k\psi(x, -y)$$

and

$$-\phi_y(x, y) + k\phi(x, y) = \phi_y(x, -y) - k\phi(x, -y)$$

We have assumed in (3.20) that $|f'| < B$ if $|z| \geq l + 1$, $y < 0$. Let us examine what this may imply for $|G|$. Let a be some point of the region (see Figure 7).

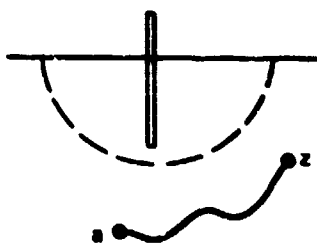


Figure 7 - Path of Integration

Then

$$f(z) = \int_a^z f'(z) dz + f(a)$$

and consequently

$$\begin{aligned} |f(z)| &< \left| \int_a^z f'(z) dz \right| + |f(a)| \\ &< B |z - a| + |f(a)| < B |z| + A \end{aligned}$$

But then

$$|G(z)| < C |z| + D, \quad C, D > 0 \quad (3.25)$$

an inequality that holds in the whole region $|z| \geq l + 1$, not just in the lower half-plane.

Near the point $z = -il$, we have assumed (3.19). Define

$$M(z) \equiv (z^2 + l^2)^{1/2} f'(z)$$

Then

$$M'(z) = \frac{z}{(z^2 + l^2)^{1/2}} f' + (z^2 + l^2)^{1/2} f''$$

or

$$(z^2 + l^2) M'(z) = z(z^2 + l^2)^{1/2} f' + (z^2 + l^2)^{3/2} f''$$

Consequently

$$\begin{aligned} (z^2 + l^2)^{3/2} G'(z) &= (z^2 + l^2)^{3/2} f'' + ik(z^2 + l^2)^{3/2} f' \\ &= (z^2 + l^2) M'(z) - z M(z) + ik(z^2 + l^2) M(z) \\ &\equiv N(z) \end{aligned}$$

Thus $(z^2 + l^2)^{3/2} G'(z)$ is analytic at the point $z = -il$ and then also through the reflection at $z = il$.

Since $(z^2 + \ell^2)^{3/2} G'(z)$ is analytic at $z = \pm i\ell$, it follows that $(z^2 + \ell^2)^{1/2} G(z)$ is also analytic at $z = \pm i\ell$. To see this, one may start with $G'(z)$ and calculate

$$\begin{aligned} G(z) - G(a) &= \int_a^z (\zeta^2 + \ell^2)^{-3/2} N(\zeta) d\zeta \\ &= \frac{1}{\ell^2} \frac{z}{(z^2 + \ell^2)^{1/2}} N(z) \Big|_a^z - \frac{1}{\ell^2} \int_a^z \frac{\zeta}{(\zeta^2 + \ell^2)^{1/2}} N'(\zeta) d\zeta \end{aligned}$$

or

$$\begin{aligned} (z^2 + \ell^2)^{1/2} G(z) &= (z^2 + \ell^2)^{1/2} \left[G(a) - \frac{1}{\ell^2} \frac{a}{(a^2 + \ell^2)^{1/2}} N(a) \right] + \frac{1}{\ell^2} z N(z) \\ &\quad - \frac{1}{\ell^2} (z^2 + \ell^2)^{1/2} \int_a^z \frac{\zeta}{(\zeta^2 + \ell^2)^{1/2}} N'(\zeta) d\zeta \end{aligned}$$

It remains to be shown that the last integral is bounded near $z = -i\ell$. This is not difficult, and we shall omit the proof here.

Let us now develop $G(z)$ in a Laurent series. Because of (3.25), the series will take the form

$$G(z) = cz + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots, \quad |z| \geq \ell + 1 \quad (3.26)$$

The condition (3.23) implies that c and all the a_n must be real. Now

$$G'(z) = c - \frac{a_1}{z^2} - \frac{2a_2}{z^3} - \dots$$

It follows easily from $\lim_{y \rightarrow -\infty} f' = 0$ that $\lim_{y \rightarrow -\infty} f'' = 0$ and hence that

$$\lim_{y \rightarrow -\infty} G'(z) = 0 \quad (3.27)$$

But then in (3.26), the constant $c = 0$.

Let us now take a circle C with center $z = 0$ and radius greater than $l + 1$ (see Figure 8). Hence the Laurent series is valid on and outside the

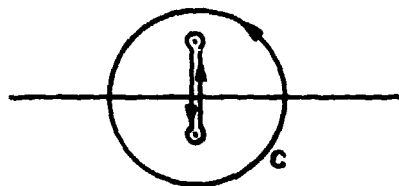


Figure 8 - Region for Laurent Series

circle and the Cauchy theorem yields

$$\oint_C G(\zeta) d\zeta = 2\pi i a_1$$

Since G is, in fact, analytic everywhere outside the flat plate, the integral along C can be deformed onto the two sides of the plate with little circles of radius ϵ around the ends, where G may have square-root singularities according to what we have shown above. Hence

$$2\pi i a_1 = \int_{-l-\epsilon+}^{l-\epsilon} G(+0 + i\eta) i d\eta + \int_{l-\epsilon}^{-l+\epsilon} G(-0 + i\eta) i d\eta$$

$$\begin{aligned}
& + \int_{-3\pi/2}^{1/2\pi} G(-i\ell + \epsilon e^{i\theta}) \epsilon e^{i\theta} i d\theta \\
& + \int_{-1/2\pi}^{3\pi/2} G(i\ell + \epsilon e^{i\theta}) \epsilon e^{i\theta} i d\theta
\end{aligned}$$

Since $G(\pm i\ell + \epsilon e^{i\theta}) \sim \epsilon^{-1/2}$, the contributions from the small circles vanish as $\epsilon \rightarrow 0$. We have from the definition of G

$$\begin{aligned}
2\pi i a_1 &= \int_{-\ell+\epsilon}^{\ell-\epsilon} [\phi_x - k\psi - i(\phi_y - k\phi)] \Big|_{-0}^{+0} i d\eta \\
&= \int_{-\ell+\epsilon}^{\ell-\epsilon} [\phi_y - k\phi] \Big|_{-0}^{+0} d\eta
\end{aligned}$$

We have been able to set $\phi_x = 0$ from (3.14). Since the plate must be a streamline, it follows that $\psi(+0, \eta) = \psi(-0, \eta) = C$ and hence that the integral of ψ vanishes. Since the left side is purely imaginary and the right side is real, we can conclude at once that both must be zero. However, we can also show that the right side is zero by invoking the antisymmetry property in (3.24). In any case we have shown that $a_1 = 0$.

Evidently

$$G'(z) = -\frac{2a_2}{z^3} - \frac{3a_3}{z^4} - \dots \quad \text{if } |z| \geq \ell + 1 \quad (3.28)$$

But then the function $(z^2 + \ell^2)^{3/2} G'(z)$ is bounded in the region $|z| \geq \ell + 1$. We have also seen that it is analytic at $z = \pm i\ell$ and hence also in the region $|z| < \ell + 1$ except possibly at $z = 0$, where there is a confluence

of boundaries with different boundary conditions. We could allow a singularity at $z = 0$ in order to represent a loss of energy in breaking of waves on the barrier. The theory would not tell us how strong to make the singularity but it would serve as a mathematical model for an energy absorber. We shall not do this, however, but instead assume that there is no loss of energy. Consequently, $(z^2 + \ell^2)^{3/2} G'(z)$ is bounded in the whole complex plane and hence, by the Liouville theorem, must be a constant:

$$G'(z) = \frac{c}{(z^2 + \ell^2)^{3/2}}, \quad c \text{ real} \quad (3.29)$$

Integrating once, we obtain

$$G(z) = \frac{C}{\ell^2} \frac{z}{(z^2 + \ell^2)^{1/2}} + d, \quad d \text{ real} \quad (3.30)$$

Up to now we have avoided specifying what we mean by $(z^2 + \ell^2)^{1/2}$. We choose the branch that behaves like z at large distances from the origin. Then on the right side of the flat plate, $(z^2 + \ell^2)^{1/2}$ takes the value $(\ell^2 - y^2)^{1/2}$ and on the left side $-(\ell^2 - y^2)^{1/2}$.

There will be no loss of generality if the constant a_0 in (3.26) is chosen to be zero for this can be accomplished by adding a suitable constant to f itself and has no effect upon the motion. With $a_0 = 0$, we see from (3.26) that $G(z)$ behaves like z^{-2} as $z \rightarrow \infty$. This behavior can be obtained in (3.30) by setting $d = -c/\ell^2$.

Hence

$$G(z) = C \left[\frac{z}{(z^2 + \ell^2)^{1/2}} - 1 \right], \quad c \text{ real} \quad (3.31)$$

This completes the first step in the reduction method, the finding of G . We must now proceed to find f .

Since

$$G = f' + ikf = e^{-ikz} \frac{d}{dz} \left[e^{ikz} f(z) \right]$$

we may immediately integrate the differential equation for f and obtain

$$f(z) = c e^{-ikz} \int^z \left[\frac{\zeta}{(\zeta^2 + l^2)^{1/2}} - 1 \right] e^{ik\zeta} d\zeta$$

In order to be specific about the constant of integration, we shall start the integral at $z = +i\infty$ and integrate along a path that lies on the right of the flat plate, as shown in Figure 9. However, in order not to lose

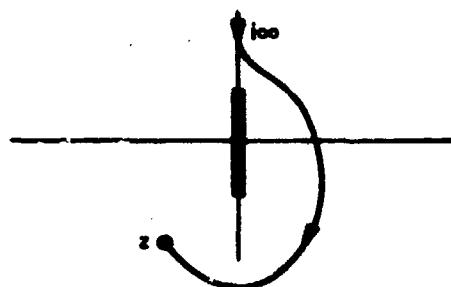


Figure 9 - Path of Integration for Equation (3.32)

generality, we must now add a constant, i.e.,

$$f(z) = c e^{-ikz} \int_{i\infty}^z \left[\frac{\zeta}{(\zeta^2 + l^2)^{1/2}} - 1 \right] e^{ik\zeta} d\zeta + B e^{-ikz} \quad (3.32)$$

In order to determine the constant B, we shall use condition (3.15).
For this purpose we must calculate

$$\begin{aligned}
 f'(z) &= -ikCe^{-ikz} \int_{-\infty}^z \left[\frac{\zeta}{(\zeta^2 + l^2)^{1/2}} - 1 \right] e^{ik\zeta} d\zeta \\
 &\quad + C \left[\frac{z}{(z^2 + l^2)^{1/2}} - 1 \right] - ikBe^{-ikz} \\
 &= -ikBe^{-ikz} + C \frac{z}{(z^2 + l^2)^{1/2}} - ikCe^{-ikz} \int_{+\infty}^z \frac{\zeta e^{ik\zeta}}{(\zeta^2 + l^2)^{1/2}} d\zeta \quad (3.33)
 \end{aligned}$$

Then, if we choose a value of y on the right side of the plate, we obtain

$$\begin{aligned}
 f'(+0 + iy) &= -ikBe^{ky} + C \frac{iy}{(l^2 - y^2)^{1/2}} - ikCe^{ky} \int_{+\infty}^{il} \frac{i\eta e^{-k\eta}}{i(\eta^2 - l^2)^{1/2}} i d\eta \\
 &\quad - ikCe^{ky} \int_{il}^{iy} \frac{i\eta e^{-k\eta}}{(l^2 - \eta^2)^{1/2}} i d\eta \\
 &= -ikBe^{ky} + iC \frac{y}{(l^2 - y^2)^{1/2}} - kCe^{ky} \int_l^\infty \frac{\eta e^{-k\eta}}{(\eta^2 - l^2)^{1/2}} d\eta \\
 &\quad + ikCe^{ky} \int_l^y \frac{\eta e^{-k\eta}}{(l^2 - \eta^2)^{1/2}} d\eta
 \end{aligned}$$

If we write $B = B_1 + iB_2$, then

$$\operatorname{Re} f'(+0 + iy) = kB_2 e^{ky} - kCe^{ky} \int_l^\infty \frac{\eta e^{-k\eta}}{(\eta^2 - l^2)^{1/2}} d\eta = 0$$

The integral is a known one, namely,

$$\int_l^\infty \frac{\eta e^{-k\eta}}{(\eta^2 - l^2)^{1/2}} d\eta = l K_1(kl)$$

where K_1 is a modified Bessel function in Watson notation.* But then

$B_2 = Cl K_1(kl)$ and we have

$$f(z) = Ce^{-ikz} \int_{+i\infty}^z \left[\frac{\zeta}{(\zeta^2 + l^2)^{1/2}} - 1 \right] e^{ik\zeta} d\zeta + [B_1 + iCl K_1(kl)] e^{-ikz} \quad (3.34)$$

where C is real (see (3.31)). This function now satisfies the free-surface condition (3.17), the condition on the plate (3.15), the sharp-corner condition (3.19), and the conditions at infinity (3.20) and (3.21). We must now bring into the picture the separate functions f_{DC} and f_{DS} , the conditions (3.16), and the radiation condition (3.18). However, before we do this, it will be useful to have the asymptotic behavior of $f(z)$ as $x \rightarrow \pm \infty$.

* See G.N. Watson, "A Treatise on the Theory of Bessel Functions," Second Edition, Cambridge, at the University Press (1944).

For this purpose we deform the path of integration as shown in Figure 10.

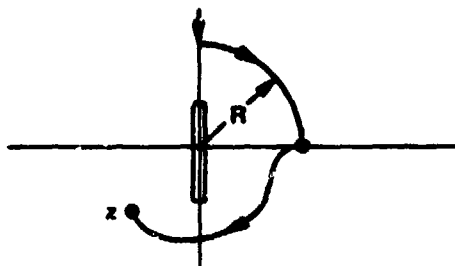


Figure 10 - Deformed Path of Integration

If we let $R \rightarrow \infty$ in the integral along the quarter-circle, it is not difficult to show that the contribution from this part converges to zero. This then gives us

$$\int_{-\infty}^z \left[\frac{\zeta}{(\zeta^2 + l^2)^{1/2}} - 1 \right] e^{ik\zeta} d\zeta = \int_{\infty}^z \left[\frac{\zeta}{(\zeta^2 + l^2)^{1/2}} - 1 \right] e^{ik\zeta} d\zeta$$

where the path of integration in the second integral must go under the plate. If we now let $x \rightarrow +\infty$, it is evident that this integral must converge to zero if it is to exist as an improper integral which it does. Hence

$$f(z) \sim [B_1 + iClK_1(kl)] e^{-ikz} \text{ as } x \rightarrow \infty \quad (3.35)$$

As $x \rightarrow -\infty$, we shall have

$$f(z) \sim [B_1 + iClK_1(kl)] e^{-ikz} - C e^{-ikz} \int_{-\infty}^{\infty} \left[\frac{\zeta}{(\zeta^2 + l^2)^{1/2}} - 1 \right] e^{ik\zeta} d\zeta$$

where the path of integration is below the plate as shown in Figure 11.

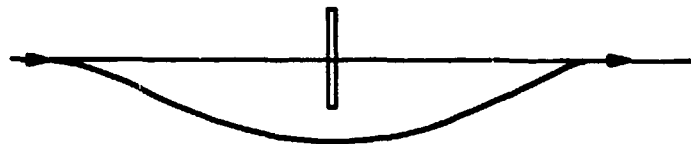


Figure 11 - Path of Integration Below the Plate

We shall complete the path of integration by adding a large semicircle in the upper half-plane, as shown in Figure 12. Again, it is easy to show that as $R \rightarrow \infty$, the contribution from the integral along the semicircle

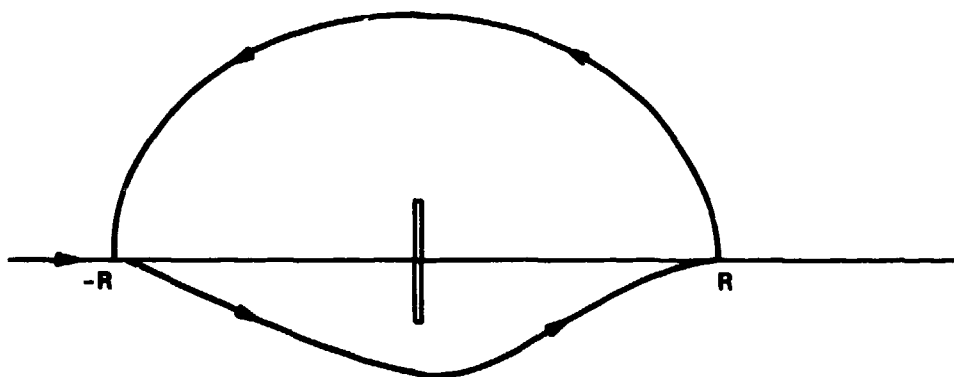


Figure 12 - Completion of Path of Integration

vanishes. Hence, adding it to the original integral has not changed its value. To be more precise, what we have shown is that

$$\int_{-\infty}^{\infty} = \int_{-R}^R + \int_{C_R}$$

where C_R is the semicircle of radius R . Since the integrand is an analytic function, we may deform the integral right onto the extended flat plate (see Figure 13). Just as in the earlier reasoning concerning G , the contributions

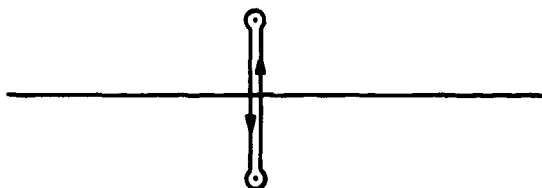


Figure 13 - Path of Integration Around Extended Flat Plate

from the little circles at the ends go to zero as their radii go to zero, and we are left with

$$\begin{aligned} \int_{-\infty}^{\infty} \left[\frac{z}{(\zeta^2 + l^2)^{1/2}} - 1 \right] e^{ik\zeta} d\zeta &= \int_{-l}^l \left[\frac{i\eta}{(l^2 - \eta^2)^{1/2}} - 1 \right] e^{-k\eta} i d\eta \\ &+ \int_l^{-l} \left[\frac{i\eta}{-(l^2 - \eta^2)^{1/2}} - 1 \right] e^{-k\eta} i d\eta = -2 \int_{-l}^l \frac{\eta e^{-k\eta}}{(l^2 - \eta^2)^{1/2}} d\eta = 2\pi l I_1(kl) \end{aligned}$$

Here again I_1 is a modified Bessel function of well known characteristics. Finally, then, we have

$$f(z) \sim [B_1 + iClK_1(kl) - 2\pi l I_1(kl)] e^{-ikz} \text{ as } x \rightarrow -\infty \quad (3.36)$$

We are now ready to invoke the radiation condition. From now on, the function f and the constants B_1 and C will all carry additional indices c or s . If we refer to (3.11) and (3.13), we see that

$$f_c(z) = -i \frac{gA}{\sigma} e^{-ikz} + f_{DC}$$

and

$$f_s(z) = - \frac{gA}{\sigma} e^{-ikz} + f_{DS}$$

Consequently, if we apply (3.35) and (3.36) to f_c and f_s ,

we find

$$\left. \begin{aligned} f_{DC}(z) &\sim \left[i \frac{gA}{\sigma} + B_{1c} + iC_c \ell K_1 \right] e^{-ikz} \\ f_{DS}(z) &\sim \left[\frac{gA}{\sigma} + B_{1s} + iC_s \ell K_1 \right] e^{-ikz} \end{aligned} \right\} \text{as } x \rightarrow +\infty$$

$$\left. \begin{aligned} f_{DC}(z) &\sim \left[i \frac{gA}{\sigma} + B_{1c} + iC_c \ell K_1 - 2\pi C_c \ell I_1 \right] e^{-ikz} \\ f_{DS}(z) &\sim \left[\frac{gA}{\sigma} + B_{1s} + iC_s \ell K_1 - 2\pi C_s \ell I_1 \right] e^{-ikz} \end{aligned} \right\} \text{as } x \rightarrow -\infty$$

(3.37)

Then the equation

$$\lim_{x \rightarrow +\infty} (f'_{DC} + kf_{DS}) = 0$$

(see (3.19)) gives

$$-ik \left[i \frac{gA}{\sigma} + B_{1c} + iC_c \ell K_1 \right] + k \left[\frac{gA}{\sigma} + B_{1s} + iC_s \ell K_1 \right] = 0$$

and the equation

$$\lim_{x \rightarrow -\infty} (f'_{DC} - kf_{DS}) = 0$$

gives

$$-ik \left[i \frac{gA}{\sigma} + B_{1c} + iC_c \ell K_1 - 2\pi C_c \ell I_1 \right] - k \left[\frac{gA}{\sigma} + B_{1s} + iC_s \ell K_1 - 2\pi C_s \ell I_1 \right] = 0$$

Separating real and imaginary parts, we find the following set of four equations for the unknown constants B_{1c} , B_{1s} , C_{1c} , C_{1s} :

$$-B_{1c} + 2\pi \ell I_1 C_c - \ell K_1 C_s = 0$$

$$-B_{1s} + \ell K_1 C_c + 2\pi \ell I_1 C_s = 0$$

$$B_{1c} - \ell K_1 C_s = 0$$

$$B_{1s} + \ell K_1 C_c = -\frac{2gA}{\sigma}$$

The solution of these equations yields

$$C_c = -\frac{K_1(k\ell)}{K_1^2 + \pi^2 I_1^2} \frac{gA}{\sigma \ell}, \quad C_s = -\frac{\pi I_1(k\ell)}{K_1^2 + \pi^2 I_1^2} \frac{gA}{\sigma \ell}$$

$$B_{1c} = \frac{-\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} \frac{gA}{\sigma}, \quad B_{1s} = \left[\frac{K_1^2}{K_1^2 + \pi^2 I_1^2} - 2 \right] \frac{gA}{\sigma}$$

Substituting into the asymptotic expressions (3.37), we find

$$f_{DC} - \frac{gA}{\sigma} = \left[\frac{\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} + i \frac{\pi^2 I_1^2}{K_1^2 + \pi^2 I_1^2} \right] e^{-ikx} \text{ as } x \rightarrow +\infty \quad (3.38)$$

$$f_{DS} \sim \frac{gA}{\sigma} \left[-\frac{\pi^2 I_1^2}{K_1^2 + \pi^2 I_1^2} - i \frac{\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} \right] e^{-ikz}, \text{ as } x \rightarrow +\infty$$

and

$$f_{DC} \sim \frac{gA}{\sigma} \left[\frac{\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} + i \frac{\pi^2 I_1^2}{K_1^2 + \pi^2 I_1^2} \right] e^{-ikz}, \text{ as } x \rightarrow -\infty$$

$$f_{DS} \sim \frac{gA}{\sigma} \left[\frac{\pi^2 I_1^2}{K_1^2 + \pi^2 I_1^2} - i \frac{\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} \right] e^{-ikz}, \text{ as } x \rightarrow -\infty$$

The sums $f_{TC} = f_{DC} + f_{IC}$ and $f_{TS} = f_{DS} + f_{IS}$ give the two parts of the transmitted wave when $x < 0$. The asymptotic expressions are as follows:

$$\left. \begin{aligned} f_{TC} &\sim \frac{gA}{\sigma} \left[\frac{\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} - i \frac{K_1^2}{K_1^2 + \pi^2 I_1^2} \right] e^{-ikz} \\ f_{TS} &\sim \frac{gA}{\sigma} \left[\frac{-K_1^2}{K_1^2 + \pi^2 I_1^2} - i \frac{\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} \right] e^{-ikz} \end{aligned} \right\} \text{ as } x \rightarrow -\infty \quad (3.39)$$

Let us use the results shown to compute ϕ_D as $x \rightarrow +\infty$, the velocity potential of the reflected wave, and $\phi_T = \phi_I + \phi_D$ as $x \rightarrow -\infty$, that of the transmitted wave:

$$\begin{aligned} \phi_D &\sim \frac{gA}{\sigma} e^{ky} \frac{1}{K_1^2 + \pi^2 I_1^2} [-\pi I_1 K_1 \cos kx + \pi^2 I_1^2 \sin kx] \cos \sigma t \\ &\quad + [-\pi^2 I_1^2 \cos kx - \pi I_1 K_1 \sin kx] \sin \sigma t \end{aligned} \quad (3.40)$$

$$= \frac{gA}{\sigma} e^{ky} \left[\frac{-\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} \cos (kx - \sigma t) + \frac{\pi^2 I_1^2}{K_1^2 + \pi^2 I_1^2} \sin (kx - \sigma t) \right], \text{ as } x \rightarrow +\infty$$

$$\phi_T \sim \frac{gA}{\sigma} e^{ky} \left[\frac{\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} \cos(kx + \sigma t) - \frac{K_1^2}{K_1^2 + \pi^2 I_1^2} \sin(kx + \sigma t) \right] \quad \text{as } x \rightarrow -\infty \quad (3.41)$$

The associated expressions for the free surface are

$$Y_D \sim A \left[\frac{\pi^2 I_1^2}{K_1^2 + \pi^2 I_1^2} \cos(kx - \sigma t) + \frac{\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} \sin(kx - \sigma t) \right], \quad x \rightarrow +\infty$$

(3.42)

and

$$Y_T \sim A \left[\frac{K_1^2}{K_1^2 + \pi^2 I_1^2} \cos(kx + \sigma t) + \frac{\pi I_1 K_1}{K_1^2 + \pi^2 I_1^2} \sin(kx + \sigma t) \right], \quad x \rightarrow -\infty$$

It is customary to introduce a reflection coefficient $R = (\text{Amplitude of } Y_D)/(\text{Amplitude of } Y_I)$ and a transmission coefficient $T = (\text{Amplitude of } Y_T)/(\text{Amplitude of } Y_I)$. Since here (see (3.9)) the amplitude of $Y_I = A$, we find

$$R = \frac{\pi I_1(kl)}{(K_1^2 + \pi^2 I_1^2)^{1/2}}, \quad T = \frac{K_1(kl)}{(K_1^2 + \pi^2 I_1^2)^{1/2}} \quad (3.43)$$

The equation $R^2 + T^2 = 1$ is simply an expression of the conservation of energy. If we had allowed a singularity at $z = 0$ to represent wave breaking, this equation would no longer hold.

We define the phase shifts by comparing the actual reflected wave with the completely reflected wave $A \cos(kx - \sigma t)$ and the transmitted wave with the incident wave $A \cos(kx + \sigma t)$:

$$Y_D = A_R \cos (kx - \sigma t - \beta_R) , x \rightarrow +\infty$$

$$Y_T = A_T \cos (kx + \sigma t - \beta_T) , x \rightarrow -\infty$$

We easily find

$$\tan \beta_R = \frac{K_1}{\pi I_1} , \quad \tan \beta_T = \frac{\pi I_1}{K_1} \quad (3.44)$$

Consequently

$$\beta_R + \beta_T = \frac{1}{2} \pi \quad (3.45)$$

The functions K_1 and I_k are well tabulated, so that there is no difficulty in plotting R , T , and β_R as functions of kl . The graphs look approximately as shown in Figure 14. It is evident that if $\lambda \ll l$, the reflection is

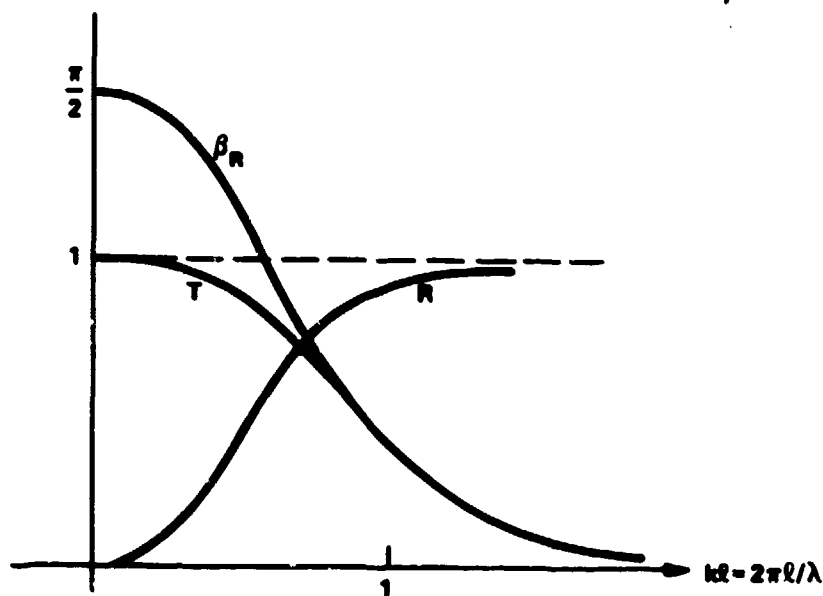


Figure 14 - R , T and β_R as Functions of kl

almost complete whereas if $\lambda \gg l$, the transmission is almost complete, results that accord with our intuition. However, comparison with experimental measurements does not show very good agreement. Part of the reason for this is probably neglect of the vortices that are formed by the flow back and forth around the sharp edge at the bottom of the plate. Their formation is not taken into account in the formulation of the problem.

We have been discussing only the asymptotic behavior of the solution when $x \rightarrow +\infty$. However, as soon as we have found B_{1c} , B_{1s} , C_c , and C_s , we may substitute into (3.34) and have the complete solution for $f_s(z)$ and $f_c(z)$. With these functions, we can then construct the solution $\phi = \phi_I + \phi_D$ for the whole region $y \leq 0$. In particular, we can find the pressure on the two sides of the plate and calculate the force and moment about 0 acting on the plate. We shall not do this to avoid getting bogged down in treating just one problem. However, this has been done by Haskind (1948); a brief discussion of the results is given in Wehausen and Laitone (1960, pp. 532-533).

PART 4: THE METHOD OF GREEN FUNCTIONS

Of the several methods that we are discussing here, the method of Green functions is certainly the most flexible in application, allowing treatment of a much wider class of problems than the other methods. However, even though it has been known for a long time and has given rise to many mathematical investigations, it did not really become important in the solution of engineering problems until perhaps the last ten years. The reason for this is that a "solution" by this method typically involved the solution of an integral equation. Although one could show in many cases that a solution to the integral equation existed, an analytic solution was usually not obtainable and a numerical solution was too difficult. The advent of high-speed computers has radically changed the situation with respect to numerical solutions, and nowadays this is almost routine.

The above remarks should not be interpreted as meaning that one cannot obtain an explicit solution by using Green functions. However, in those situations where one can do this one can, as far as I am aware, also solve the problems by another method. For example, Havelock derived the Michell integral by using a Green function. However, Michell himself derived it from Fourier analysis, having first derived a representation theorem analogous to (2.22).

We shall approach this method by way of some Green identities, which, in turn, are directly derivable from the Gauss divergence theorem. Let ϕ and ψ be any two functions defined in a certain three-dimensional region V and having second derivatives there, including the boundary S . Then one of the Green identities states the following:

$$\int_V [\psi \Delta \phi - \phi \Delta \psi] dV = \int_S [\psi \phi_n - \phi \psi_n] dS \quad (4.1)$$

where the normal vector points out of the region V . If in addition both ϕ and ψ satisfy the Laplace equation, the left-hand side vanishes and

$$\int_S [\psi \phi_n - \phi \psi_n] dS = 0 \quad (4.2)$$

A particular solution of the Laplace equation in three dimensions, in either $P = (x, y, z)$ or $Q = (\xi, \eta, \zeta)$, is

$$\psi = \frac{1}{r}, \quad r = [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2} \quad (4.3)$$

The only exception is at the point $P = Q$, where the solution is singular. Let us take Q as the integration variable. We must then exclude a small region containing P from the integration. Let this be a sphere V_ϵ of radius ϵ with surface S_ϵ . Then (4.1) takes the form

$$\begin{aligned} \int_{V-V_\epsilon} \left[\frac{1}{r} \Delta \phi - \phi \Delta \frac{1}{r} \right] dV &= \int_S \left[\frac{1}{r} \phi_n - \phi \frac{\partial}{\partial n} \frac{1}{r} \right] dS(Q) \\ &+ \int_{S_\epsilon} \left[-\frac{1}{\epsilon} \phi_r - \phi \frac{1}{\epsilon^2} \right] \epsilon^2 d\Omega \end{aligned}$$

Since the left-hand side vanishes identically and since the integral over S_ϵ converges to $-4\pi \phi(P)$ as $\epsilon \rightarrow 0$, we find another identity:

$$\phi(P) = \frac{1}{4\pi} \int_S \left[\phi_v \frac{1}{r} - \phi(Q) \frac{\partial}{\partial v} \frac{1}{r} \right] dS \quad (4.4)$$

Here we have written v instead of n in order to make clear that the normal derivative is taken in the variables (ξ, η, ζ) .

If the point P is taken on the surface S at a point where the surface is smooth, we may show that (4.4) still holds if 4π is replaced by 2π .

We may now generalize (4.4) further by exploiting (4.2). If we add to $1/r$ any harmonic function ψ (i.e., one satisfying the Laplace equation and having no singularities in V), (4.4) will still hold. We shall choose a special kind of function to add to $1/r$. Let $H(x, y, z, \xi, \eta, \zeta)$ be harmonic in each set of variables (x, y, z) and (ξ, η, ζ) . We define

$$G(P, Q) = \frac{1}{r} + H(P, Q) \quad (4.5)$$

It then follows from (4.4) and (4.2) that

$$\phi(P) = \frac{1}{4\pi} \int_S [\phi_{,v}(Q)G(P, Q) - \phi G_{,v}] dS(Q) \quad (4.6)$$

As before, if the point P is taken on S , the 4π is replaced by 2π .

We have been considering only three-dimensional motion. However, there are corresponding theorems if the motion is two dimensional. In this case (4.5) is replaced by

$$G(P, Q) = \log \frac{1}{r} + H(P, Q) \quad (4.7)$$

where $P = (x, y)$, $Q = (\xi, \eta)$ and $r = [(x-\xi)^2 + (y-\eta)^2]^{1/2}$. Formula (4.6) becomes

$$\phi(P) = \frac{1}{2\pi} \int_C [\phi_{,v}G - \phi G_{,v}] ds \quad (4.8)$$

where C is a contour bounding a two-dimensional region S and ds is arc length. If P is on C at a point where C is smooth, then the 2π is replaced by π .

Let us now turn to several problems where we can exploit (4.6) or (4.8).

FORCED HARMONIC MOTION

Let us suppose that a body is being forced to undergo periodic motion of frequency σ and in such a way that there is no average displacement. We shall denote the average position by S_0 . Let the Oy -axis pass through (or near) the body and let Σ_R be a cylinder of radius R , large enough to contain the body. We further denote the portion of the free surface F

inside Σ_R by F_R and the portion of the bottom B inside Σ_R by B_R (see Figure 15).

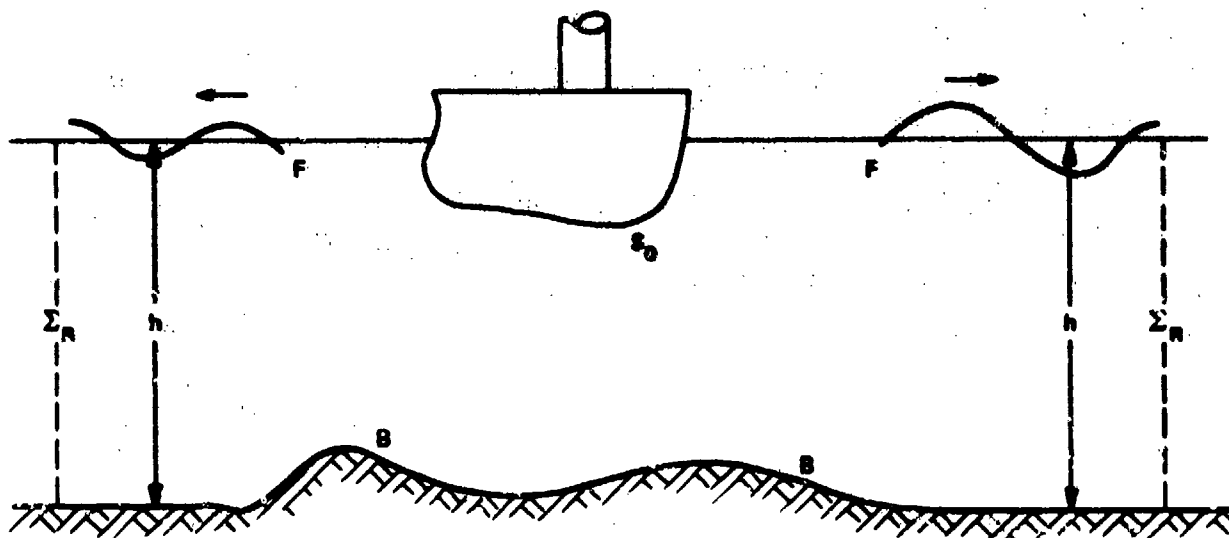


Figure 15 - Representation of the Problem of Forced Harmonic Motion

If we write

$$\Phi(x, y, z, t) = \phi_1(x, y, z) \cos \sigma t + \phi_2 \sin \sigma t = \text{Re } \phi e^{-i\sigma t} \quad (4.9)$$

where $\phi = \phi_1 + i\phi_2$, then from (1.13), the free-surface boundary condition

$$\phi_y(x, 0, z) - v\phi = 0, \quad v = \sigma^2/g \quad (4.10)$$

The condition on the bottom is

$$\phi_n|_B = 0 \quad (4.11)$$

and the condition on S_0 is

$$\phi_n|_{S_0} = v_n \quad (4.12)$$

Since this is still in terms of Φ rather than ϕ , let us decompose V_n :

$$\begin{aligned}
 v_n(x,y,z,t) &= v_{n1} \cos \sigma t + v_{n2} \sin \sigma t \\
 &= \operatorname{Re} V_n(x,y,z) e^{-i\sigma t}
 \end{aligned}
 \tag{4.13}$$

where $V_n(x,y,z) = v_{n1} + i v_{n2}$. Usually the fact that the body displacements are small is exploited further and we can write

$$\begin{aligned}
 v_n(x,y,z,t) &= \dot{x}_1(t) n_1 + \dot{y}_1(t) n_2 + \dot{z}_1(t) n_3 \\
 &\quad + \dot{\alpha}(\vec{r} \times \vec{n})_1 + \dot{\beta}(\vec{r} \times \vec{n})_2 + \dot{\gamma}(\vec{r} \times \vec{n})_3
 \end{aligned}
 \tag{4.14}$$

Here $(x_1(t), y_1(t), z_1(t))$ describe the translation displacements, $(\alpha(t), \beta(t), \gamma(t))$ describe the angular displacements of the body, and \vec{r} is a vector from the origin to a point (x, y, z) on the body surface S_0 . If we define

$$\begin{aligned}
 \alpha_1 &= x_1, \alpha_2 = y_1, \alpha_3 = z_1 \\
 \alpha_4 &= \beta z - \gamma y, \alpha_5 = \gamma x - \alpha z, \alpha_6 = \alpha y - \beta x
 \end{aligned}
 \tag{4.15}$$

we may write (4.14) in the form

$$v_n(x,y,z,t) = \sum_{k=1}^6 \dot{\alpha}_k(t) r_k(x,y,z)
 \tag{4.16}$$

Now $\alpha_k(t)$ itself can be written as

$$\alpha_k(t) = \alpha_k' \cos \sigma t + \alpha_k'' \sin \sigma t = \operatorname{Re} \alpha_k e^{-i\sigma t}
 \tag{4.17}$$

where $\alpha_k = \alpha_k' + i\alpha_k''$. (Whether α_k is a function of t or a complex amplitude should be clear from context.) Next ϕ is decomposed as follows:

$$\phi(x,y,z,t) = \sum_k \dot{\alpha}_k(t) \phi_1^{(k)}(x,y,z) + \sigma \alpha_k(t) \phi_2^{(k)}(x,y,z)
 \tag{4.18}$$

The boundary condition (4.12) now gives

$$\phi_{1n}^{(k)}|_{S_0} = n_k, \quad \phi_{2n}^{(k)}|_{S_0} = 0 \quad (4.19)$$

If we wish to divide ϕ into sine and cosine components, we may write it as follows:

$$\phi = \sum_k \sigma [\alpha_k'' \phi_1^{(k)} + \alpha_k' \phi_2^{(k)}] \cos \sigma t + \sum_k \sigma [-\alpha_k' \phi_1^{(k)} + \alpha_k'' \phi_2^{(k)}] \sin \sigma t$$

so that in the notation of (4.9)

$$\begin{aligned} \phi_1 &= \sum_k \sigma [\alpha_k'' \phi_1^{(k)} + \alpha_k' \phi_2^{(k)}] \\ \phi_2 &= \sum_k \sigma [-\alpha_k' \phi_1^{(k)} + \alpha_k'' \phi_2^{(k)}] \end{aligned} \quad (4.20)$$

To justify the decomposition (4.18), we should demonstrate that the problem has a unique solution. It is possible to do this after we add one more condition, the radiation condition. In the two examples treated by the Fourier method, we satisfied this by manipulating some undetermined constants. However, when treating the diffraction about a plate by the reduction method, we formulated the radiation condition directly in terms of f_{DC} and f_{DS} (see (3.18)). There is an analogous formulation for the present case. If, analogously to (4.9), we write $\phi^{(k)} = \phi_1^{(k)} + i\phi_2^{(k)}$, then the radiation condition for $\phi^{(k)}$ takes the following form:

$$\begin{aligned} \lim_{R \rightarrow \infty} R^{1/2} [\phi_R^{(k)} - ik_0 \phi^{(k)}] &= 0, \\ \sigma^2 &= gk_0 \tanh k_0 h \end{aligned} \quad (4.21)$$

where $R = [x^2 + z^2]^{1/2}$. The radiation condition for $\phi = \phi_1 + i\phi_2$ is the same. It is a consequence of (4.21), which we shall not prove here, that

$$\phi^{(k)} = O(R^{-1/2}) \text{ as } R \rightarrow \infty \quad (4.22)$$

i.e., that $R^{1/2}|\phi^{(k)}| < M < \infty$ if $R > R_0$ for some sufficiently large R_0 .

As far as boundary conditions are concerned, the only difference between ϕ and the $\phi^{(k)}$ is in the boundary condition on S_0 . All of them satisfy (4.10), (4.11), and (4.21). However, $\phi^{(k)}$ satisfies (4.19) whereas ϕ satisfies

$$\phi_n|_{S_0} = V_n(x,y,z) \quad (4.23)$$

Much of the following can apply to either one. We shall write out the development for ϕ . To convert to $\phi^{(k)}$, we need only replace $V_n(x,y,z)$ by $n_k(x,y,z)$.

Let us now apply (4.6) to the domain bounded by S_0 , F_R , B_R , and Σ_R . We find the following:

$$\begin{aligned} 4\pi\phi(P) &= \int_{S_0 \cup F_R \cup B_R \cup \Sigma_R} [\phi_\nu(Q) G(P,Q) - \phi(Q) G_\nu(P,Q)] dS(Q) \\ &= \int_{S_0} [V_\nu(Q) G(P,Q) - \phi(Q) G_\nu(P,Q)] dS \\ &\quad + \int_{F_R} \left[\frac{\sigma^2}{g} \phi(\xi, 0, \zeta) G(P; \xi, 0, \zeta) - \phi G_\eta \right] d\xi d\zeta \\ &\quad + \int_{B_R} -\phi(Q) G_\nu(P,Q) dS \\ &\quad + \int_{\Sigma_R} [\phi_R^G - \phi G_R] R d\theta d\eta \end{aligned} \quad (4.24)$$

where we have used (4.23) in the first integral, (4.10) in the second, and (4.11) in the third.

We now consider several cases.

Case 1. We begin by writing the last integral of (4.24) as follows:

$$\int_{\Sigma_R} \{R^{1/2} [\phi_R - ik_0\phi] R^{1/2}G - R^{1/2} [G_R - ik_0G] R^{1/2}\phi\} d\theta d\gamma \quad (4.25)$$

In this integral, we know from (4.21) and (4.22) that $R^{1/2}[\phi_R - ik_0\phi] \rightarrow 0$ and that $R^{1/2}\phi$ remains bounded as $R \rightarrow \infty$. Let us now suppose that it is possible in (4.5) to find a function H such that

$$G_\eta(P; \xi, 0, \zeta) - vG = 0$$

$$G_v(P, Q) = 0 \text{ for } Q \text{ on } B \quad (4.26)$$

$$\lim_{R \rightarrow \infty} R^{1/2}[G_R(P; \xi, \eta, \zeta) - ik_0G] = 0, \quad R = (\xi^2 + \zeta^2)^{1/2}$$

Any consequences we may draw from these properties depend, of course, on our being able to construct such a function.

It is now immediately evident that the integrals over F_R and B_R are zero. In the last integral, in the form (4.25), we see that now $R^{1/2}[G_R - ik_0G] \rightarrow 0$ and $R^{1/2}G$ remains bounded as $R \rightarrow \infty$. Consequently the integral over Σ_R converges to zero as $R \rightarrow \infty$. This leaves us then with only the integral over S_0 . If we now let P be a point of S_0 instead of a point in the interior, the 4π is replaced by 2π and we may write the resulting equation in the following form:

$$\phi(P) + \frac{1}{2\pi} \int_{S_0} \phi(Q) G_v(P, Q) dS(Q) = \frac{1}{2\pi} \int_{S_0} v_v(Q) G(P, Q) dS(Q) \quad (4.27)$$

This is an integral equation for the determination of $\phi(P)$ for $P \in S_0$. If this equation has been solved, then (4.24), which takes the form below, gives $\phi(P)$ for any point inside the fluid:

$$\phi(P) = \frac{1}{4\pi} \int_{S_0} [V_v(Q)G(P,Q) - \phi(Q)G_v(P,Q)] dS \quad (4.28)$$

Once we have found $\phi(P)$, we can also find other quantities of interest. For example, the free surface is given by

$$Y(x,z,t) = -\frac{1}{g} \phi_t(x,0,z,t) = \frac{\sigma}{g} \operatorname{Re} i\phi(x,0,z)e^{-i\sigma t} \quad (4.29)$$

The force acting on the body (i.e., the hydrodynamic part) is given by

$$\begin{aligned} \vec{F} &= \int_{S_0} p \vec{n} dS = -\rho \int_{S_0} \phi_t \vec{n} dS \\ &= \sum_{k=1}^6 -\rho \int [\ddot{\alpha}_k(t)\phi_1^{(k)} + \sigma \dot{\alpha}_k(t)\phi_2^{(k)}] \vec{n} dS \end{aligned}$$

or, in components

$$\begin{aligned} F_i &= -\ddot{\alpha}_k \rho \int_{S_0} \phi_1^{(k)} n_i dS - \dot{\alpha}_k \sigma \rho \int_{S_0} \phi_2^{(k)} n_i dS \\ &= -\ddot{\alpha}_k \mu_{ik} - \dot{\alpha}_k \lambda_{ik} \end{aligned} \quad (4.30)$$

where μ_{ik} and λ_{ik} are often respectively called "added masses" and "damping coefficients." In this case they are functions of the frequency as well as the body geometry. There are some advantages for further developments in replacing n_i in (4.30) by $\phi_{in}^{(k)}$, according to (4.50). However, we shall not explore this further.

Under what conditions can one find a Green function that satisfies (4.26)? It is possible to construct such a function for either infinite or constant finite depth by applying either the Fourier method or the reduction method. The functions can be found in several places, e.g., in

Wehausen and Laitone (1960), for both two and three dimensions. We shall later consider the case when the bottom is not of constant depth.

The development of a computational method based upon integral equations has been carried out for two-dimensional motion by Werner Frank (1967) of NSRDC and almost simultaneously by Lebreton (1967).

Case II. Let us try the very simple choice

$$G(P,Q) = \frac{1}{r} \quad (4.31)$$

In this case the integrals over F_R and B_R will not vanish. Let us examine the integral over Σ_R as given in (4.25). We know from (4.21) that $R^{1/2}[\phi_R - ik_0\phi]$ can be made as small as we wish by taking R large enough. Can we assert that $R^{1/2}G = R^{1/2}/r$ remains bounded? Let us write

$$R_1 = (x^2 + z^2)^{1/2}, \quad R = (\xi^2 + \zeta^2)^{1/2} \quad (4.32)$$

and let α be the angle between the vectors (x,z) and (ξ,ζ) . Then

$$r = [R^2 + R_1^2 - 2RR_1 \cos \alpha + (y - \eta)^2]^{1/2} \quad (4.33)$$

Evidently

$$\frac{R^{1/2}}{r} = \frac{R^{-1/2}}{\left[1 + \left(\frac{R_1}{R}\right)^2 - \frac{2R_1}{R} \cos \alpha + \left(\frac{y-\eta}{R}\right)^2\right]^{1/2}} \quad (4.34)$$

If P is taken near Q , it is evident that the denominator becomes small and hence that $R^{1/2}/r$ becomes large. Properly we should examine the integral

$$I = R^{1/2} \int_{-h}^0 d\eta \int_0^{2\pi} d\theta \frac{1}{r}$$

We can show, for example, that

$$I < \frac{4h}{R^{1/2}} K\left(\frac{R_1}{R}\right)$$

where $K(p)$ is the complete elliptic integral. We can undoubtedly do better than this. In any case, it is not enough, for $K(p) \rightarrow \infty$ as $p \rightarrow 1$. We shall approach the treatment of the integral over Σ_R somewhat differently.

In discussing the Fourier method, we found solutions of the form (2.17). If in the solution for x_0 we had introduced polar coordinates

$$x = R \cos \theta, \quad z = R \sin \theta \quad (4.35)$$

we should have found the following solutions:

$$x_0 = \begin{pmatrix} J_n(k_0 R) \\ Y_n(k_0 R) \end{pmatrix} \times \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} \quad (4.36)$$

where n is an integer and J_n and Y_n are Bessel functions in the Watson notation. If we consider the region exterior to some bounded body, we find that the only contributions for x_1 are exponentially decreasing in R and that the part of a solution representing outgoing waves has the following form:

$$\phi \sim \sum_{n=0}^{\infty} a_n \cosh[k_0(y+b)] \cos(n\theta + \delta_n) [J_n(k_0 R) + iY_n] \quad (4.37)$$

where, as usual, $J_n + iY_n = H_n^{(1)}$. The combination is usually called the Hankel function of the first kind. Its asymptotic expansion for large $k_0 R$ is well known, and we may write the asymptotic expression for ϕ as follows:

$$\phi = \sum_{n=0}^{\infty} a_n \cosh[k_0(y+h)] \cos(n\theta + \delta_n) \left(\frac{2}{\pi k_0 R}\right)^{1/2} x \quad (4.38)$$

$$\left\{ \exp i(k_0 R - \frac{n\pi}{2} - \frac{\pi}{4}) \left(1 - \frac{n^2 - \frac{1}{4}}{24k_0 R}\right) + O\left(\frac{1}{(k_0 R)^2}\right) \right\}$$

A straightforward computation yields

$$\phi_R = \frac{-1}{2R} \phi + ik_0 \phi + O((k_0 R)^{-5/2}) \quad (4.39)$$

We now substitute this expression for ϕ_R into the integral over Σ_R in its original form in (4.24), neglecting the term of $O((k_0 R)^{-3/2})$. After some reordering of (4.24), we can write it in the following form:

$$4\pi\phi(P) + \int_{S_0} \phi(Q) G_v(P, Q) dS + \int_{F_R} \phi(Q) \left[G_\eta(P, Q) - \frac{\sigma^2}{2} G \right] dS \quad (4.40)$$

$$+ \int_{B_R} \phi(Q) G_v(P, Q) dS + \int_{\Sigma_R} \phi(Q) \left[\left(ik_0 - \frac{1}{2R} \right) \frac{1}{r} - \frac{\partial}{\partial R} \frac{1}{r} \right] dS$$

$$= \int_{S_0} v_v(Q) G(P, Q) dS$$

If we now let P approach any boundary, we have the same equation with 4π replaced by 2π . This is then an integral equation for the determination of $\phi(P)$ for P on $S_0 \cup F_R \cup B_R \cup \Sigma_R$. Once it has been determined, the equation above determines ϕ at any interior point. We should keep in mind that in deriving Equation (4.40), an approximation has been made in evaluating the integral over Σ_R .

This method has been used for numerical calculations in both two and three dimensions by R.W.-C. Yeung (1973; see also Bai and Yeung, 1974). In

comparing this procedure with the earlier one in which we used a Green function satisfying (4.26), the following points should be kept in mind:

1. The Green function satisfying (4.26) is much more complicated than (4.31). However, this is offset by the fact that ϕ is solved only for P on S_0 . In the Yeung method one finds ϕ for P on $S_0 \cup F \cup B_R \cup \Sigma_R$, which means, of course, a much larger matrix in the discrete version of the integral equation.
2. Although in the Yeung method we have required a flat bottom for R large enough, it does not need to be flat in the region near the body. Furthermore, in the two-dimensional version, the flat region can have different depths on the two sides. This did not seem to be possible with the more complicated Green function satisfying (4.26). We now turn our attention to this question.

Case III. We now suppose that our Green function satisfies (4.26) except that $G_v = 0$ for Q on B is replaced by

$$G_\eta(P; \xi, -h, \zeta) = 0 \quad (4.41)$$

We have mentioned earlier that it is possible to construct such a Green function. This Green function will now be applied to the situation indicated schematically in Figure 15. However, we shall suppose that the part of the bottom that is not at depth h is limited to a finite stretch that we shall denote by B_0 (see Figure 16).

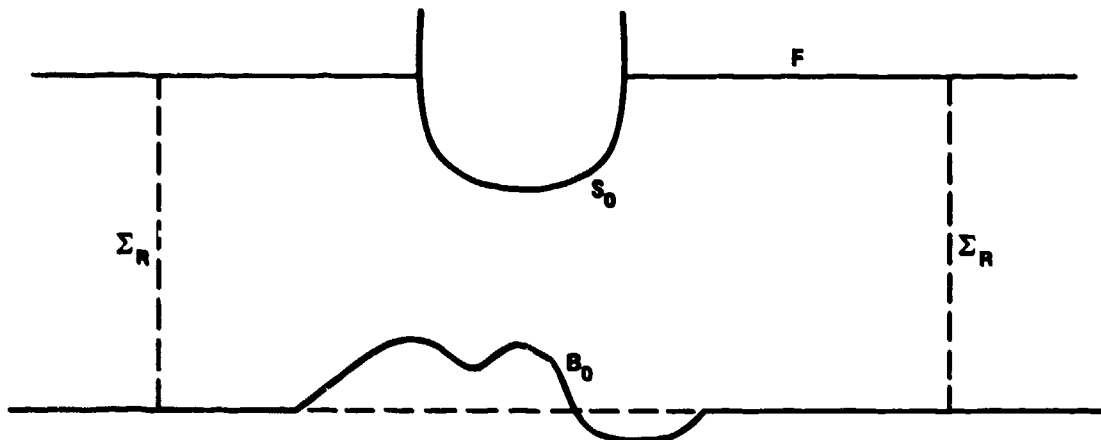


Figure 16 - Limitation on the Part of the Bottom Not at Depth h

If we now use this Green function in (4.24) and take the limit $R \rightarrow \infty$, we obtain the following equation:

$$4\pi\phi(P) = \int_{S_0} [V_v(Q)G(P,Q) - \phi(Q)G_v] dS + \int_{B_0} -\phi(Q)G_v(P,Q) dS \quad (4.42)$$

If we now let P be a point of either S_0 or B_0 , we obtain the following integral equation:

$$\phi(P) + \frac{1}{2\pi} \int_{S_0 \cup B_0} \phi(Q)G_v(P,Q) dS = \frac{1}{2\pi} \int_{S_0} V_v(Q)G(P,Q) dS \quad (4.43)$$

This is almost exactly the same as the integral equation (4.26) except that the integral on the left-hand side is now over $S_0 \cup B_0$ instead of just S_0 .

Now by comparing Cases I, II, and III when the bottom is uneven near the body, we may make the following observation. Case I is not feasible for we do not know how to construct the Green function. Both Cases II and III are feasible. Case II has a simple Green function, but we must find $\phi(P)$ over an extended boundary. Case III has a more complicated Green function, but we must solve for $\phi(P)$ over a more restricted boundary. Some comparisons of computer times for the case of a completely flat bottom are given in a paper by Bai and Yeung (1974). In the case considered there, a two-dimensional one, about twice as much time was required for the Yeung method (Case II) as for the Frank method (Case I).

DIFFRACTION PROBLEMS AND THE HASKIND RELATIONS

The following remarks are a sort of appendix to Case III discussed above. With the same geometry considered there, let us suppose that the

body is fixed but that there is a known incident wave with velocity potential ϕ_I . If ϕ is the velocity potential for the total fluid motion, then the diffracted wave is defined as usual by

$$\phi = \phi_I + \phi_D \quad (4.44)$$

The diffraction potential must, of course, satisfy the free-surface condition and radiation conditions. In addition, it must satisfy the boundary condition

$$\phi_{Dn} = -\phi_{In} \text{ on both } S_0 \text{ and } B_0 \quad (4.45)$$

The force acting upon the body as a result of the presence of the fluid motion is given by

$$\begin{aligned} F_i &= \int_{S_0} p n_i dS = -\rho \int_{S_0} (\phi_{It} + \phi_{Dt}) n_i dS \\ &= \text{Re} \left\{ -i\sigma\rho \int_{S_0} (\phi_I + \phi_D) n_i dS e^{-i\sigma t} \right\} \end{aligned} \quad (4.46)$$

The Haskind relations as usually presented for a horizontal bottom or infinite depth, allow solution of the diffraction problem to be avoided if the forced-motion problem has already been solved. Here we shall show that this is also true if the bottom is uneven.

Instead of the function ϕ defined by the integral equation (4.43), we shall use the function $\phi_1^{(k)}$ defined by the same equation with $V_v(Q)$ replaced by $n_k(Q)$. Then $\phi_1^{(k)}$ satisfies the boundary condition (4.19) and in addition

$$\phi_{1n}^{(k)} = 0 \text{ on } B \quad (4.47)$$

We suppose that this function has been found.

Consider now the volume bounded by $S_0 \cup F_R \cup B_R \cup \Sigma_R$. We apply the Green identity (4.2) to ϕ_{Dt} and $\phi_1^{(k)}$:

$$\begin{aligned} 0 &= \int_{S_0 \cup F_R \cup B_R \cup \Sigma_R} [\phi_D \phi_{1n}^{(k)} - \phi_{Dn} \phi_1^{(k)}] dS \\ &= \int_{S_0 \cup B_0 \cup \Sigma_R} [\phi_D \phi_{1n}^{(k)} - \phi_{Dn} \phi_1^{(k)}] dS \end{aligned} \quad (4.48)$$

for both ϕ_{Dn} and $\phi_{1n}^{(k)}$ equal zero on the flat part of B (recall that $\phi_{1n} = 0$ there). Also, $\phi_{1n}^{(k)} = 0$ on B_0 . Since both ϕ_D and $\phi_1^{(k)}$ satisfy the radiation condition, the integral over Σ_R tends to zero as $R \rightarrow \infty$. Hence we may write the last equation in the following form:

$$\begin{aligned} \int_{S_0} \phi_D n_K dS &= \int_{S_0} \phi_D \phi_{1n}^{(k)} dS = \int_{S_0 \cup B_0} \phi_{Dn} \phi_1^{(k)} dS \\ &= - \int_{S_0 \cup B_0} \phi_{In} \phi_1^{(k)} dS \end{aligned} \quad (4.49)$$

Then Equation (4.46) may be rewritten as:

$$F_1 = \text{Re} \left\{ -i\sigma\rho \left[\int_{S_0} (\phi_I \phi_{1n}^{(k)} - \phi_{In} \phi_1^{(k)}) dS - \int_{B_0} \phi_{In} \phi_1^{(k)} dS \right] e^{i\sigma t} \right\} \quad (4.50)$$

F_1 is now expressed completely in terms of known functions. The expression (4.50) differs from the usual one only in the presence of the integral over B_0 . This poses no problem in principle for in solving (4.43), we have found $\phi_1^{(k)}$ on both S_0 and B .

DIFFRACTION ABOUT ISLANDS, HARBOR OSCILLATION

As a sort of diversion, I should like to consider a slightly different type of Green-function problem, one that is two dimensional in the mathematical sense. We begin by considering the physical situation shown in Figure 17:

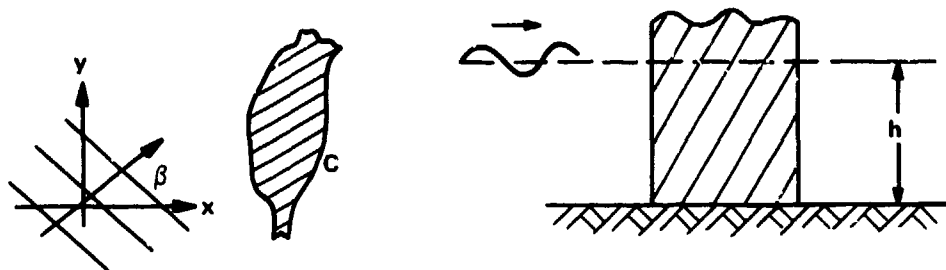


Figure 17 - Representation of the Physical Problem of Diffraction about Islands

The bottom is assumed to be horizontal and the island to have vertical walls. As a consequence of this simplified geometry, the problem may be treated as two dimensional. The incoming wave will have the velocity potential

$$\begin{aligned}\phi_I(P,t) &= \frac{gA}{\sigma \cosh k_0 h} \cosh k_0(z+h) \sin [k_0(x \cos \beta + y \sin \beta) - \sigma t] \\ &= \text{Re } \phi_I(P) e^{-i\sigma t}, \quad \sigma^2 = gk_0 \tanh k_0 h\end{aligned}\tag{4.51}$$

and the surface profile

$$Y_I(x,y,t) = A \cos[k_0(x \cos \beta + y \sin \beta) - \sigma t]\tag{4.52}$$

Note that here we have taken z vertical (see Figure 18).

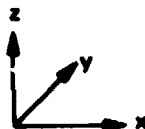


Figure 18 - References Axes for Diffraction about Islands

The diffraction potential $\phi_D(P, t) = \text{Re } \phi_D(P) e^{-i\sigma t}$ will satisfy the free-surface condition, the radiation condition, the bottom condition, and

$$\phi_{Dn}|_C = -\phi_{In}|_C \quad (4.53)$$

From the form of ϕ_D , we can see that it is possible to precipitate z out of the problem by recognizing that ϕ_D must have the form

$$\phi_D(x, y, z) = \hat{\phi}(x, y) \frac{\cosh k_0(z + h)}{\cosh k_0 h} \quad (4.54)$$

Then ϕ_D automatically satisfies the free-surface and bottom conditions. The Laplace equation becomes

$$\Delta_3 \phi_D = [\hat{\phi}_{Dxx} + \hat{\phi}_{Dyy} + k_0^2 \hat{\phi}_D] \frac{\cosh k_0(z + h)}{\cosh k_0 h} = 0 \quad (4.55)$$

or

$$\Delta_2 \hat{\phi}_D + k_0^2 \hat{\phi}_D = 0$$

This is known as the Helmholtz equation. Henceforth we shall drop the circumflexes $\hat{}$.

As a preliminary step, we go back to one of the Green identities in two dimensions.

$$\int_S (u \Delta v - v \Delta u) dS = \int_C (uv_n - vu_n) ds$$

If both u and v satisfy the Laplace equation, the left-hand side obviously vanishes. However, if both u and v satisfy the Helmholtz equation, the left-hand side also vanishes. Hence

$$\int_C (uv_n - vu_n) dS = 0$$

Now suppose that it is possible to find a solution of Helmholtz equation of the form

$$v = \log \frac{1}{r} + F(x, y; \xi, \eta)$$

$$r = [(x - \xi)^2 + (y - \eta)^2]^{1/2}$$

where $P = (x, y)$ is a fixed point and $Q = (\xi, \eta)$ is any other point in S or on C (see Figure 19). Of course, v has a singularity as $(\xi, \eta) \rightarrow (x, y)$ so

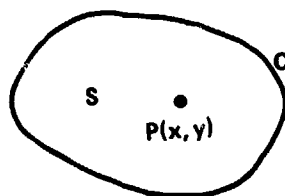


Figure 19 - Region S for Helmholtz Equation

that, just as in the case of the Laplace equation, we must exclude a small circle C_ϵ of radius ϵ about P in order to apply the equation above

$$\int_C + \int_{C_\epsilon} [uv_n - vu_n] dS = 0$$

The same limiting procedure that we used earlier to derive (4.8) now leads to

$$u(P) = \frac{1}{2\pi} \int_C (vu_n - uv_n) dS(Q), \quad P \in S_{\text{interior}} \quad (4.56)$$

We return now to the island problem. Consider the region bounded by C and a large circle of radius R, Σ_R (see Figure 20). We hope that we can construct a function

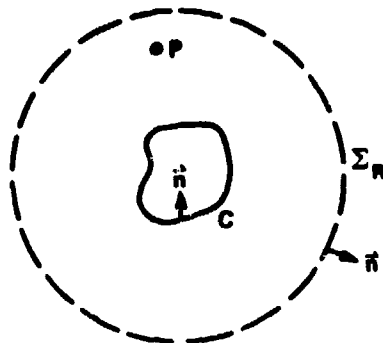


Figure 20 - Region Bounded by C and Σ_R

$$\begin{aligned} G(P, Q) &= \log \frac{1}{r} + F \\ &= \log \frac{1}{r} + F_1 + iF_2 \end{aligned} \quad (4.57)$$

such that G satisfies both the Helmholtz equation and the radiation condition

$$\lim_{r \rightarrow \infty} \sqrt{r} (G_r - iK_0 G) = 0 \quad (4.58)$$

We shall write, according to the formula we have just derived,

$$\phi_D(P) = \frac{1}{2\pi} \int_C [\phi_{D\nu} G - \phi_D G_\nu] dS + \frac{1}{2\pi} \int_{\Sigma_R} [\phi_{D\nu} G - \phi_D G_\nu] dS$$

Since both ϕ_D and G satisfy the radiation condition, we can show as we did earlier that $\lim_{R \rightarrow \infty} \int_{\Sigma_R} = 0$. Hence

$$\phi_D(P) = \frac{1}{2\pi} \int [\phi_{D\nu}(Q) G(P, Q) - \phi_D G_\nu] dS$$

or, using the boundary condition on C,

$$\phi_D(P) + \frac{1}{2\pi} \int_C \phi_D(Q) G_v(P, Q) dS = - \frac{1}{2\pi} \int_C \phi_{Iv}(Q) G(P, Q) dS \quad (4.59)$$

If the point P is taken on the boundary C, then, as before, the 2π is replaced by π :

$$\phi_D(P) + \frac{1}{\pi} \int_C \phi_D(Q) G_v(P, Q) dS = - \frac{1}{\pi} \int_C \phi_{Iv}(Q) G(P, Q) dS \quad (4.60)$$

This is an integral equation for ϕ_D and the remaining problem, aside from numerical computation, is to determine whether the required function G can be constructed.

We shall simply give the function rather than actually construct it. It is possible to show that

$$G(P, Q) = i \frac{\pi}{2} [J_0(k_0 r) + iY_0(k_0 r)] = i \frac{\pi}{2} H_0^{(1)}(k_0 r) \quad (4.61)$$

The integral equation for ϕ_D is then

$$\phi_D(P) + \frac{1}{2} \int_C \phi_D(Q) H_{0v}^{(1)} dS = - \frac{1}{2} \int_C \phi_{Iv} H_0^{(1)}(k_0 r) dS \quad (4.62)$$

Once ϕ_D has been found on the boundary C, then we can use the more general equation (4.59) to find $\phi_D(P)$ for P outside C:

$$\phi_D(P) = - \frac{1}{4} \int_C [\phi_D(Q) H_{0v}^{(1)}(k_0 r) + \phi_{Iv}(Q) H_0^{(1)}(k_0 r)] dS \quad (4.63)$$

The above results are all well known in the theory of diffraction of acoustic and electromagnetic waves upon infinitely long cylinders.

Let us now consider a slightly different geometrical configuration. Figure 21 is supposed to represent a coastline with a harbor and an incident wave. We again suppose the bottom to be at depth h and all coastlines to be vertical cliffs.

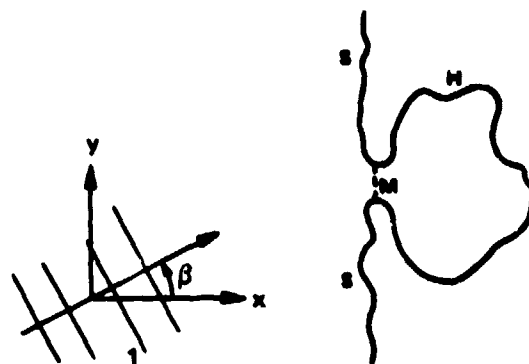


Figure 21 - Incident Wave on a Coastline with Harbor

We shall follow a procedure of J.-J. Lee (1971) in dealing with this problem. At the mouth of the harbor, place a rigid fictitious wall M and solve the problem in two parts. First we find the solution in the region outside the harbor with the wall M in place, i.e.,

$$\phi_{Dn}|_{S \cup M} = -\phi_{In}|_{S \cup M} \quad (4.64)$$

In a simplified model of the shore line, it might be a straight line (see Figure 22). In this case, the diffracted wave ϕ_D would simply be the

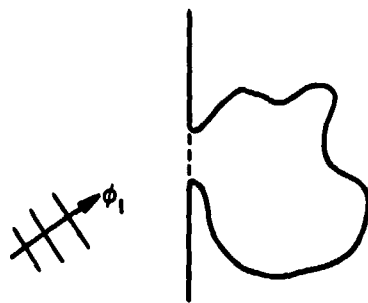


Figure 22 - Straight-Line Coast

plane wave reflected from the shore line. Note that in this case, the diffracted wave does not satisfy the same radiation condition used earlier.

However, with the configuration shown in Figure 23, the earlier radiation condition applies. $\phi_I + \phi_D$ cannot really give the solution outside the

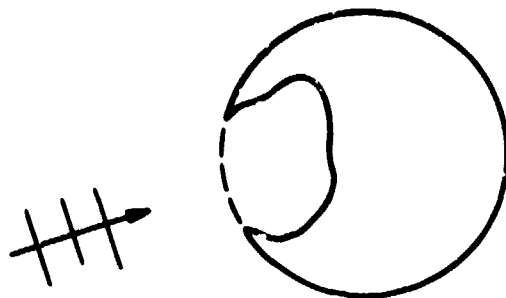


Figure 23 - Circular Coast

harbor for the effect of the harbor has been neglected. The incident wave ϕ_I will certainly excite some sort of motion inside the harbor and, in turn, this motion will excite some motion outside the harbor that will radiate away from the mouth. Hence the motion in the exterior will be represented by

$$\phi_{\text{ext}} = \phi_I + \phi_D + \phi_R$$

where ϕ_R satisfies the radiation condition (4.58). For a point P in the exterior, we then know that

$$\phi_R = \frac{1}{4} \int_{M \cup S} [\phi_{Rv} H_0^{(1)} - \phi_R H_{0v}^{(1)}] dS \quad (4.65)$$

If we let P be on the boundary and remember that $\phi_{Rn}|_S = 0$, we may write this as follows:

$$\phi_R(P) + \frac{1}{2} \int_{M \cup S} \phi_R H_{0v}^{(1)} dS = \frac{1}{2} \int_M \phi_{Rv} H_0^{(1)} dS \quad (4.66)$$

We shall denote the velocity potential inside the harbor by ϕ_H . By the Green theorem, we have for any point P inside the harbor

$$\phi_H(P) = \frac{1}{4} \int_{M \cup H} [\phi_{H\nu}^{(1)} - \phi_{H0\nu}^{(1)}] dS \quad (4.67)$$

where we have used the Green function that satisfies the radiation condition because we want to match ϕ_H with ϕ_{ext} . Since $\phi_{Hn}|_H = 0$, part of the integral vanishes. Letting P be on the boundary, we find

$$\phi_H(P) + \frac{1}{2} \int_{M \cup H} \phi_{H0\nu}^{(1)} dS = \frac{1}{2} \int_M \phi_{H\nu}^{(1)} dS \quad (4.68)$$

Our matching conditions are the following:

$$\phi_{\text{ext}} = \phi_I + \phi_D + \phi_R = \phi_H \text{ on } M \quad (4.69)$$

$$\phi_{\text{ext } n} = \phi_{Rn} = -\phi_{Hn} \text{ on } M$$

(since the positive direction of n is opposite for ϕ_R and ϕ_H)

Before applying these in the above equation, it will be convenient to introduce some new notation. Let us define the following operations:

$$I\psi = \psi$$

$$H_0^{(1)A}\psi = \frac{1}{2} \int_A \psi H_0^{(1)} dS \quad (4.70)$$

$$H_{0\nu}^{(1)A}\psi = \frac{1}{2} \int_A \psi H_{0\nu}^{(1)} dS$$

Then the integral equations (4.66) and (4.58) can be written as follows:

$$(I + H_{0\nu}^{(1)M \cup S})\phi_R = H_0^{(1)M}\phi_{Rn} \quad (4.71)$$

$$(I + H_{0\nu}^{(1)M \cup H})\phi_H = H_0^{(1)M}\phi_{Hn}$$

Let us denote the values of ϕ_{Rn} and ϕ_{Hn} on M by

$$\phi_{Hn}(P) = -\phi_{Rn}(P) \equiv F(P), P \in M \quad (4.72)$$

Then for a point P of M , we may invert the above equations and write

$$\begin{aligned} \phi_R &= -(I + H_{0v}^{(1)M \cup S})^{-1} H_0^{(1)M} F \\ \phi_H &= (I + H_{0v}^{(1)M \cup H})^{-1} H_0^{(1)M} F \end{aligned} \quad (4.73)$$

Since $\phi_H = \phi_I + \phi_D + \phi_R$ on M , we have

$$(I + H_{0v}^{(1)M \cup H})^{-1} H_0^{(1)M} F = -(I + H_{0v}^{(1)M \cup S})^{-1} H_0^{(1)M} F + \phi_I + \phi_D$$

or

$$H_0^{(1)M} F = [(I + H_{0v}^{(1)M \cup H})^{-1} + (I + H_{0v}^{(1)M \cup S})^{-1}]^{-1} (\phi_I + \phi_D) \quad (4.74)$$

This gives $H_0^{(1)M} F$ in terms of $\phi_I + \phi_D$. We could proceed to find F , but this is not necessary, for having found $H^{(1)M} F$, we can now use Equations (4.70) to find ϕ_R or ϕ_H on $M \cup S$ or $M \cup H$, respectively. Once we know ϕ_R and ϕ_H on these boundaries, we can find them at any point of their respective domains. For example, to find $\phi_H(P)$ for any P inside the harbor, we use

$$\phi_H(P) = -\frac{1}{2} H_{0v}^{(1)M \cup H} \phi_H + \frac{1}{2} H_0^{(1)M} F \quad (4.75)$$

The operator notation has been introduced for the following reason. If we are treating the exact problem, then these are integral operators acting on continuous functions. However, if we discretize the problem by looking for ϕ_H , say, only at discrete points P_1, P_2, \dots on the boundary,

we then replace the integral operator by a matrix operator and the continuous function by a vector. Nevertheless, the equations in operator form remain the same. Thus use of the operator notation introduces a certain economy in thought.

The method has been successfully tried by Lee (1971) and in a more elaborate version by Lee and Rnichlen (1972).

AN INITIAL-VALUE PROBLEM FOR A WAVEMAKER

The last example that we shall consider differs from the preceding ones in two respects: we use a time-dependent Green function and we are able to give an explicit solution. The problem under consideration is nothing but the initial-value problem for a wavemaker in a wall. It is thus closely related to the problem considered on page 26.

We suppose that some section of the wall $x = 0$ is flexible and moves in some predetermined way

$$x = F(y, z, t) , (y, z) \in S_0 \quad (4.76)$$

We may take the bottom to be given by

$$y = -H(x, z) \quad (4.77)$$

but eventually we take the fluid infinitely deep in order to give an explicit solution. The appropriate boundary and initial conditions for the problem are:

$$\begin{cases} \phi_x(0, y, z, t) = \begin{cases} F_t(y, z, t) , & (y, z) \in S_0 \\ 0 , & (y, z) \notin S_0 \end{cases} \\ \phi_n(x, -H(x, z), z, t) = 0 , & h = H(x, z) \\ \lim_{y \rightarrow \infty} \phi_y = 0 , & h = \infty \end{cases} \quad (4.78)$$

$$\begin{aligned}
\phi_{tt}(x,0,z,t) + g\phi_y &= 0 \\
\phi(x,y,z,0) = \phi_t(x,0,z,0) &= 0 \\
|\phi|, |\phi_t|, |\nabla\phi|, \text{ and } |\nabla\phi_t| &\text{ are bounded}
\end{aligned}
\tag{4.78}$$

(Cont'd)

Since Equation (4.6) did not involve time, it may still be applied to this problem. However, we wish to make an efficient choice of the Green function. We shall suppose that it is possible to construct a function $G = 1/r + H(P,Q,t)$ satisfying the following conditions:

$$\begin{aligned}
G_\xi(P;0,\eta,\zeta;t) &= 0 \\
\begin{cases} G_\nu(P;\xi, -H(\xi,\zeta),\zeta;t) = 0 \text{ if } h = H(x,z) \\ \lim_{\eta \rightarrow \infty} G_\eta = 0, \text{ if } h = \infty \end{cases} \\
G_{tt}(P;\xi,0,\zeta;t) + gG_\eta &= 0 \\
\begin{cases} G(P;Q;t) = G(P;Q; -t) \\ \text{or } G_t(P;Q;0) = 0 \end{cases} \\
G(P;\xi,0,\zeta;0) &= 0 \\
G = O(R^{-2}), G_R = O(R^{-3}), G_y = O(R^{-3})
\end{aligned}
\tag{4.79}$$

We shall now apply (4.6) to the fluid bounded by a large vertical circular cylinder Σ_R of radius r , and the parts of the wall W , the free surface F , and the bottom B included inside Σ_R . However, we shall apply (4.6) to ϕ_t rather than ϕ . Furthermore, we shall take the time variable in ϕ to be τ and in G to be $t - \tau$. This does not, of course, invalidate the use of (4.6). We then have the following:

$$\begin{aligned}
4\pi\phi_t(P;\tau) &= \int_W [-\phi_{t\xi}(0,\eta,\zeta;\tau)G(P;0,\eta,\zeta;t-\tau) + \phi_t G_\xi] d\eta d\zeta \\
&+ \int_F [\phi_{t\eta}(\xi,0,\zeta;\tau)G(P;\xi,0,\zeta;t-\tau) - \phi_t G_\eta] d\xi d\zeta
\end{aligned}$$

where we have already used the boundary conditions on B and the asymptotic condition for large R to eliminate the integrals over B and Σ_R . We may also exploit the boundary conditions on W and F to write this equation as follows:

$$\begin{aligned}
4\pi\phi_t(P;\tau) &= - \int_{S_0} F_{tt}(\eta,\zeta;\tau)G(P;0,\eta,\zeta;t-\tau)d\eta d\zeta \\
&+ \frac{1}{g} \int_F [-\phi_{ttt}(\xi,0,\zeta;\tau)G(P;\xi,0,\zeta;t-\tau) + \phi_t G_{tt}]d\xi d\zeta \\
&= - \int_{S_0} F_{tt}(\tau)G(t-\tau)d\eta d\zeta \\
&- \frac{1}{g} \int_F \frac{\partial}{\partial \tau} [\phi_{tt}(\tau)G(t-\tau) + \phi_t(\tau)G_t(t-\tau)]d\xi d\zeta \\
&= - \int_{S_0} F_{tt}(\tau)G(t-\tau)d\eta d\zeta \\
&+ \frac{\partial}{\partial \tau} \int_F [\phi_\eta(\tau)G(t-\tau) - \frac{1}{g} \phi_t(\tau)G_t(t-\tau)]d\xi d\zeta
\end{aligned} \tag{4.80}$$

We now integrate this equation from 0 to t:

$$\begin{aligned}
4\pi\phi(P;t) - 4\pi\phi(P;0) &= - \int_0^t d\tau \int_{S_0} F_{tt}(\tau)G(t-\tau)d\eta d\zeta \\
&+ \int_F [\phi_\eta(t)G(0) - \frac{1}{g} \phi_t(t)G_t(0)]d\xi d\zeta \\
&- \int_F [\phi_\eta(0)G(t) - \frac{1}{g} \phi_t(0)G_t(t)]d\xi d\zeta
\end{aligned} \tag{4.81}$$

If we now make use of the initial conditions for G and of Equations (1.9) and (1.10), we may write this equation as follows:

$$4\pi\phi(P;t) = 4\pi\phi(P;0) - \int_0^t d\tau \int_{S_0} F_{tt}(\tau)G(t-\tau)d\eta d\zeta \quad (4.82)$$

$$- \int_F [Y_t(\xi, \zeta, 0)G(P; \xi, 0, \zeta; t) + Y(\xi, \zeta, 0)G_t(P; \xi, 0, \zeta; t)]d\xi d\zeta$$

In fact, we have assumed that $\phi(P;0) = 0$ and $\phi_r(\xi, 0, \zeta; 0) = 0$ as initial conditions for ϕ . Hence (4.82) becomes

$$4\pi\phi(P,t) = - \int_0^t d\tau \int_{S_0} F_{tt}(\eta, \zeta, \tau)G(P; 0, \eta, \zeta; t-\tau)d\eta d\zeta \quad (4.83)$$

This solves the problem formally, but we are left with the problem of finding a Green function satisfying (4.79). This function can be constructed for either $H(x,z) = h = \text{const}$ or $h = \infty$. We give it here only for $h = \infty$. We give first the result without the condition $G_\xi(P; 0, \eta, \zeta; t) = 0$:

$$G_0(P, Q, t) = \frac{1}{r} - \frac{1}{r_1} + 2 \int_0^\infty dk e^{k(y+\eta)} J_0(kR) [1 - \cos(\sqrt{gkt})] \quad (4.84)$$

$$= \frac{1}{r} + \frac{1}{r_1} - 2 \int_0^\infty dk e^{k(y+\eta)} J_0(kR) \cos(\sqrt{gkt})$$

where

$$r = [(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]^{1/2}$$

$$r_1 = [(x - \xi)^2 + (y + \eta)^2 + (z - \zeta)^2]^{1/2}$$

$$R = [(x - \xi)^2 + (z - \zeta)^2]^{1/2}$$

In order to construct a Green function that also satisfies $G_\xi(P; 0, \eta, \zeta; t) = 0$, we use the method of images. Define

$$\begin{aligned}
r_2 &= [(x + \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]^{1/2} \\
r_3 &= [(x + \xi)^2 + (y + \eta)^2 + (z - \zeta)^2]^{1/2} \\
R_2 &= [(x + \xi)^2 + (z - \zeta)^2]^{1/2}
\end{aligned}$$

Then

$$\begin{aligned}
G(P; Q; t) &= \frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \\
&- 2 \int_0^\infty dk e^{k(y+\eta)} [J_0(kR) + J_0(kR_2)] \cos [\sqrt{gk}\tau]
\end{aligned} \tag{4.85}$$

If we substitute this into (4.83), we obtain

$$\begin{aligned}
4\pi\Phi(P; t) &= - 2 \int_{S_0} d\eta d\zeta \left[\frac{1}{\sqrt{x^2 + (y - \eta)^2 + (z - \zeta)^2}} \right. \\
&\quad \left. + \frac{1}{\sqrt{x^2 + (y + \eta)^2 + (z - \eta)^2}} \right] \int_0^t F_{tt}(\eta, \zeta; t) d\tau \\
&+ 4 \int_{S_0} d\eta d\zeta \int_0^\infty dk e^{k(y+\eta)} J_0(k[x^2 + (z - \zeta)^2]^{1/2}) \int_0^t F_{tt}(\eta, \zeta; t) \\
&\quad \cos \sqrt{gk}(t - \tau) d\tau
\end{aligned} \tag{4.86}$$

If we make the special choice (2.38),

$$F(y, z, t) = F(y, z) \sin \sigma t$$

the integrals with respect to τ can be evaluated. An interesting problem is then to find the asymptotic form of the solution as $t \rightarrow \infty$. This should

and does agree with the form obtained by the Fourier-integral method. The asymptotic form may be written as follows:

$$\begin{aligned} \Phi_{\infty}(P;t) = \frac{\sigma}{2\pi} \int_{S_0} d\eta d\zeta F(\eta, \zeta) & \left\{ \left[\frac{1}{\sqrt{x^2 + (y - \eta)^2 + (z - \zeta)^2}} \right. \right. \\ & \left. \left. + \frac{1}{\sqrt{x^2 + (y + \eta)^2 + (z - \zeta)^2}} \right] \sin \sigma t + 2k_0 \sin \sigma t \int_0^{\infty} \frac{dk}{k - k_0} e^{k(y+\eta)} \right. \\ & \left. J_0(kR) - 2\pi k_0 \cos \sigma t e^{k_0(y+\eta)} J_0(k_0 R) \right\} \end{aligned} \quad (4.87)$$

The use of time-dependent Green functions in the manner shown above is apparently first due to Volterra (1934). It has been further exploited by Finkelstein (1957), Wehausen (1967), and W.-C. Lin (1966). Many people prefer to work entirely with frequency by initially taking Fourier transforms (or Laplace transforms).

PART 5: THE METHOD OF MULTIPOLES

The method of multipoles can be used in either two or three dimensions and consists essentially in adding together singularities of higher and higher order with intensities to be determined by the boundary conditions. In the case discussed here, the singularities are chosen to satisfy the free-surface and radiation conditions. The intensities are then selected to satisfy the boundary condition on an oscillating body.

The method occurs in various guises, but in the type of problem we consider, it was apparently first used by Ursell (1949), who treated the heaving motion of a circular cylinder. The analogous problem for a heaving sphere was later considered by Havelock (1955). Others who have used it are Tasai (1959), Porter (1960), and C.M. Lee (1968), but the list is not exhaustive.

The present approach to the method differs somewhat from the usual one, but as will be shown later, the results are the same. The present approach seems to have the advantage of showing from the beginning that the sum of singularities usually employed really is sufficient to solve the problem. Once again the method is illustrated by the problem of the forced harmonic motion of a body in a free surface. The treatment will be two dimensional and will have much in common with that of diffraction from a vertical plate.

FORCED HARMONIC MOTION

Figure 24 shows schematically the physical situation. We shall

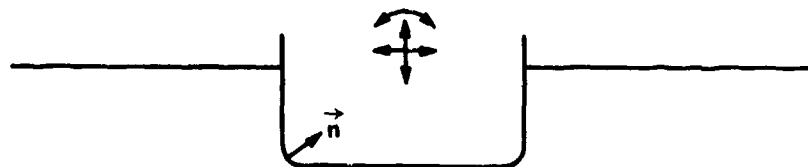


Figure 24 - Oscillating Body on the Free Surface

suppose that there is a complex velocity potential of the form

$$F(z,t) = f_c \cos \sigma t + f_s \sin \sigma t \quad (5.1)$$

From page 39 we may conclude immediately that both f_c and f_s satisfy

$$\text{Im}\{f'(x+i0) + ik_0 f\} = 0, \quad k_0 = \sigma^2/g \quad (5.2)$$

and

$$\lim_{x \rightarrow \pm\infty} [f'_c \pm k_0 f_s] = 0 \quad (5.3)$$

We further suppose that

$$\lim_{y \rightarrow -\infty} f' = 0 \quad (5.4)$$

and that

$$|f'| < B \quad \text{if} \quad |z| \geq a, \quad y \leq 0 \quad (5.5)$$

where $|z| = a$ contains the cylinder. Let the surface of the cylinder be described by $z(s) = x(s) + iy(s)$, where s is arc length measured from some convenient point. Then

$$\begin{aligned} f'(z(s)) z'(s) &= (\phi_x - i\phi_y)(x' + iy') \\ &= \phi_x x' + \phi_y y' - i(-\phi_x y' + \phi_y x') \\ &= \phi_{\text{tang}} - i\phi_n \end{aligned} \quad (5.6)$$

Hence the final boundary condition is

$$\text{Im} f'(z(s)) z'(s) = -V_n \quad (5.7)$$

We now proceed almost exactly as in the case of the diffraction about a vertical plate, i.e., by the reduction method. Define

$$g(z) = f' + ik_0 f \quad (5.8)$$

Then, repeating the steps taken in that problem except for considerations concerning the singularity at $z = -i\ell$, we find

$$g(z) = cz + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots, \quad |z| \geq a \quad (5.9)$$

where c and a_n are real. We may also conclude from (5.4), just as in the diffraction problem, that $c = 0$. Furthermore, a_0 may be absorbed into the stream function without loss of generality, so that we finally have

$$g(z) = \sum_{n=1}^{\infty} \frac{a_n}{z^n}, \quad |z| \geq a \quad (5.10)$$

One of the important properties of power series representing analytic functions is that they converge absolutely up to the nearest singularity. Since the function g must be analytic everywhere outside the body, we know that (5.10) must converge at least up to a circle just containing the body, as shown in an example given in Figure 25. If the body is a semicircle,

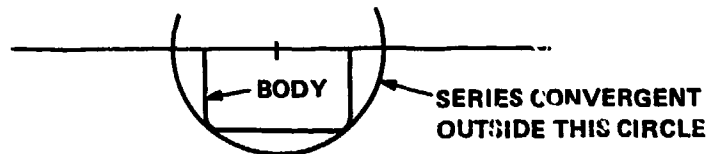


Figure 25 - Region of Convergence

then the series (5.10) will converge right up to the body. Moreover, since there is no singular behavior of g on the circle representing the body, the radius of convergence must be smaller than the radius of the body itself. It will also be true that the radius of convergence of (5.10) for a non-circular body with smooth boundary will be smaller than the circle just embracing the body. However, this is no guarantee that it will hold right up to the boundary of the body. Thus, at the moment, we seem to be constrained to treat only a body of circular section with center in the free surface. For the time being we shall do this and later consider what to do if the cylindrical section is not circular.

We now integrate (5.10) and obtain

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-ik_0 z} \int_{\infty}^z \frac{e^{ik_0 \zeta}}{\zeta^n} d\zeta + b_0 e^{-ik_0 z} \quad (5.11)$$

where $b_0 = b_0' + ib_0''$, but the a_n are real. The path of integration is taken from $z = +\infty$ and lies below the body (see Figure 26).

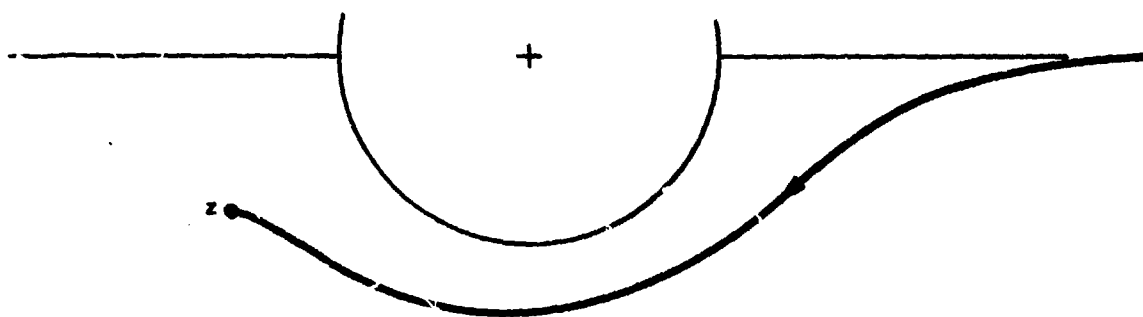


Figure 26 - Path of Integration for (5.11)

We define

$$f_n(z) = e^{-ik_0 z} \int_{\infty}^z \frac{e^{ik_0 \zeta}}{\zeta^n} d\zeta \quad (5.12)$$

An easy integration by parts shows that

$$f_n = \frac{1}{ik_0} \left[\frac{1}{z^n} + n f_{n+1} \right]$$

or

$$ik_0 f_n - n f_{n+1} = \frac{1}{z^n}, \quad n=1, 2, 3, \dots \quad (5.13)$$

Furthermore, it is easy to establish that

$$f_n'(z) = -ik_0 f_n + \frac{1}{z^n} \quad (5.14)$$

so that

$$f_{n+1} = -\frac{1}{n} f_n' = (-1)^n \frac{1}{n!} f_1^{(n)} \quad (5.15)$$

Let us now consider only the infinite-sum part of f_c or f_g :

$$\begin{aligned} \sum_{n=1}^{\infty} a_n f_n &= a_1 f_1 + \frac{a_2}{ik_0} [ik_0 f_2 - 2f_3] + \\ &+ \left[\frac{2a_2}{ik_0} + a_3 \right] \frac{1}{ik_0} [ik_0 f_3 - 3f_4] + \\ &+ \left[\frac{3}{ik_0} \left(a_3 + \frac{2a_2}{ik_0} \right) + a_4 \right] \frac{1}{ik_0} [ik_0 f_4 - 4f_5] + \dots \quad (5.16) \\ &= a_1 f_1 + \frac{a_2}{ik_0} \frac{1}{z^2} + \left[\frac{a_3}{ik_0} + 2 \frac{a_2}{(ik_0)^2} \right] \frac{1}{z^3} + \\ &+ \left[\frac{a_4}{ik_0} + 3 \frac{a_3}{(ik_0)^2} + 3.2 \frac{a_2}{(ik_0)^3} \right] \frac{1}{z^4} + \dots \\ &= b_1 f_1 + b_2 \frac{1}{z^2} + b_3 \frac{1}{z^3} + \dots \end{aligned}$$

where b_1 is real, b_2 is imaginary and b_3, b_4, \dots are complex. The last series is, however, misleading, for it does not make evident that the b_n must be related if the free-surface condition is to be satisfied. This series may be arranged as follows:

$$\sum_{n=1}^{\infty} a_n f_n = a_1 f_1 + \frac{a_2}{ik_0} \left[\frac{1}{z^2} + \frac{2}{ik_0} \frac{1}{z^3} \right] + \frac{a_3}{ik_0} \left[\frac{1}{z^3} + \frac{3}{ik_0} \frac{1}{z^4} \right] +$$

$$+ \left[\frac{a_4}{ik_0} + 3.2 \frac{a_2}{(ik_0)^3} \right] \left[\frac{1}{z^4} + \frac{4}{ik_0} \frac{1}{z^5} \right] + \dots \quad (5.17)$$

Now each coefficient after the first is purely imaginary, and it is possible to show that each term individually satisfies the free-surface condition and radiation condition. Define

$$m_n(z) = \frac{1}{z^{n+1}} + i \frac{k_0}{n} \frac{1}{z^n} \quad (5.18)$$

Then it is easy to confirm that

$$\text{Im}\{m_n'(x+10) + ik_0 m_n\} = 0$$

Hence, $b_n m_n(z)$, b_n real, also satisfies this.

We assert that we may write

$$\sum a_n f_n = b_1 f_1 + \sum_{n=2}^{\infty} b_n m_n(z), \quad b_1 = a_1, b_2, b_3, \dots \text{ real} \quad (5.19)$$

of course, the manipulations leading to (5.17) do not prove this, but once the combination defining m_n has been recognized, it is not difficult to prove (5.19). For example, we may do this by using the relationship

$$f_n = -\frac{n}{k_0^2} [m_n + (n+1) f_{n+2}]$$

Let us now consider

$$f_1 = e^{-ik_0 z} \int_{\infty}^z \frac{e^{ik_0 \zeta}}{\zeta} d\zeta$$

$$= \int_{\infty}^z \frac{e^{ik_0(\zeta-z)}}{\zeta} d\zeta$$

We shall make the change of variables

$$k_0(\zeta-z) = -kz \quad \text{or} \quad k = k_0 \frac{\zeta-z}{-z}, \quad \zeta = -\frac{k-k_0}{k_0} z \quad (5.20)$$

Formally, this gives immediately

$$f_1 = \int_{\infty}^0 \frac{e^{-ikz}}{k-k_0} dk = -\int_0^{\infty} \frac{e^{-ikz}}{k-k_0} dk$$

where the path of integration is still to be determined. Let us take our path in the ζ -plane as shown in Figure 27 (two cases are shown).

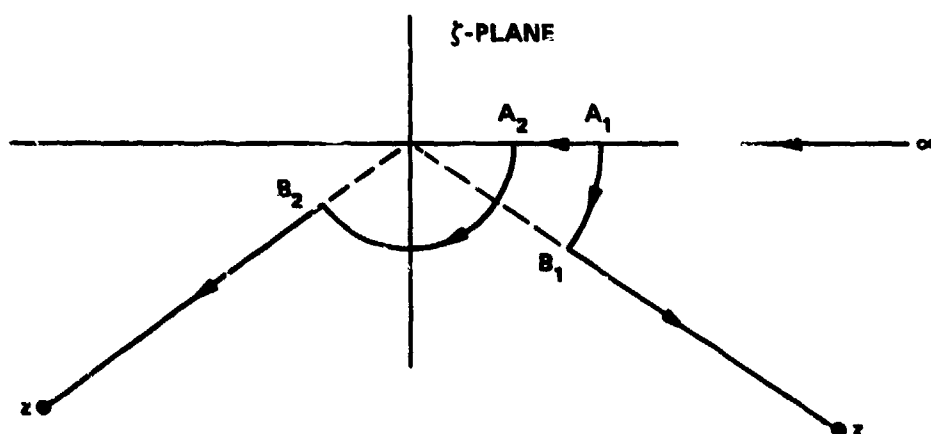


Figure 27 - Paths of Integration in ζ -Plane

The transformed paths in the k -plane will then appear as shown in Figure 28. The new paths are a rotation and stretching of the original paths.

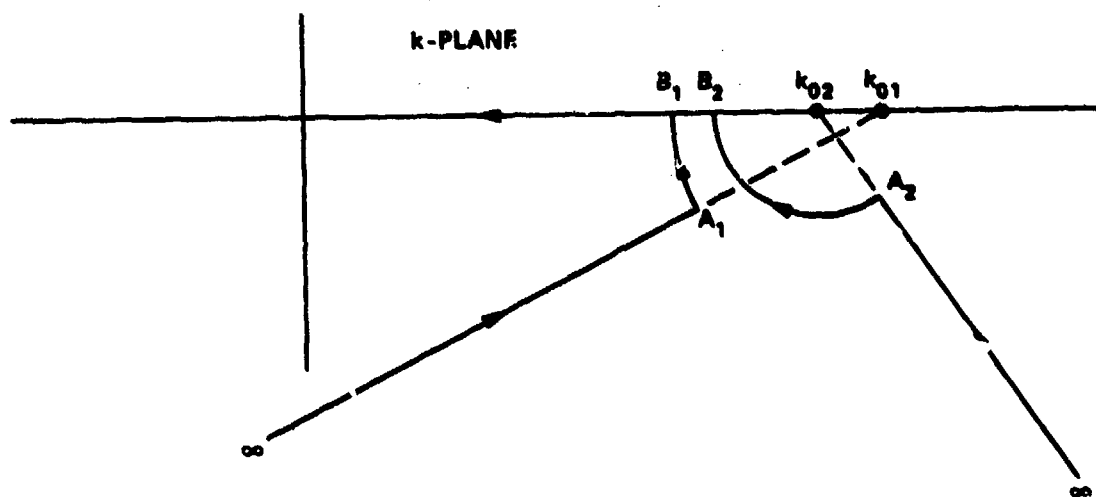


Figure 28 - Transformed Paths in the k -Plane

Let us now examine the arcs Σ_R joining the path to ∞ with the positive x -axis (see Figure 29).

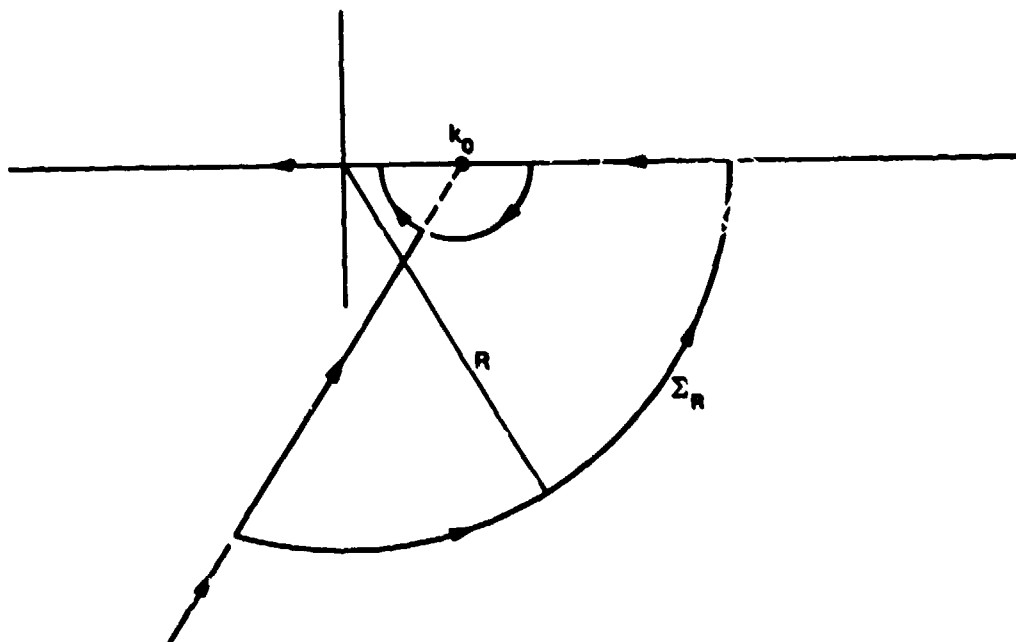


Figure 29 - Arcs Σ_R Joining the Path to ∞

We find easily that the contribution of Σ_R goes to zero as $R \rightarrow \infty$. Hence we may deform the path to one along the x-axis except for dodging around k_0 (see Figure 30).



Figure 30 - Path along the x-Axis

If we choose the path around k_0 as a semicircle of radius ϵ and then take the limit as $\epsilon \rightarrow 0$, we will obtain a Cauchy principal-value integral plus a half residue:

$$f_1(z) = - \int_0^{\infty} \frac{e^{-ikz}}{k-k_0} dk - \pi i e^{-ik_0 z} \quad (5.21)$$

Now consider the radiation condition (5.3). In order to satisfy this, we must find the asymptotic behavior of f as $x \rightarrow \pm \infty$. If we use the series (5.19) as part of the representation, it is obvious that we may discard the contribution of the terms in m_n as $x \rightarrow \pm \infty$. Hence we need to examine the asymptotic behavior only of

$$a_1 f_1(z) + b_0 e^{-ik_0 z} \quad (5.22)$$

It is evident that

$$f(z) \sim b_0 e^{-ik_0 z} \quad \text{as } x \rightarrow +\infty \quad (5.23)$$

If we write (5.22) in the form

$$a_1 e^{-ik_0 z} \left[\int_{-\infty}^z - \int_{-\infty}^{\infty} \right] \frac{e^{ik_0 \zeta}}{\zeta} d\zeta + b_0 e^{-ik_0 z}$$

we see that

$$f(z) \sim -a_1 e^{-ik_0 z} \int_{-\infty}^{\infty} \frac{e^{ik_0 \zeta}}{\zeta} d\zeta + b_0 e^{-ik_0 z} \text{ as } x \rightarrow -\infty$$

As in the case discussed on page 50, where the path in Figure 11 has been replaced by that in Figure 12, we may replace the integral from $-\infty$ to $+\infty$ by the closed path shown in Figure 31.

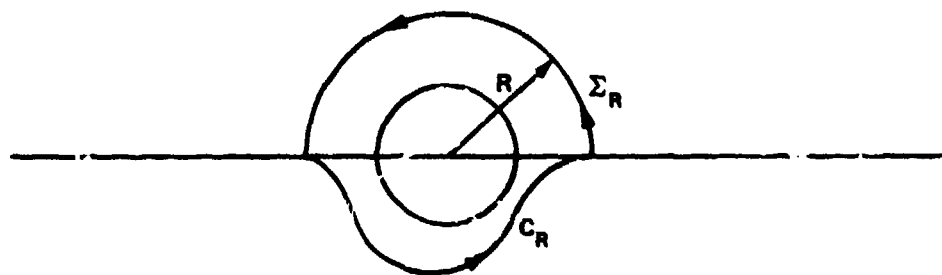


Figure 31 - Equivalent Path of Integration for $f(z)$

$$\int_{-\infty}^{\infty} = \int_{C_R} + \int_{\Sigma_R}$$

But then we know from the Cauchy Integral theorem that

$$\int_{-\infty}^{\infty} \frac{e^{ik_0 \zeta}}{\zeta} d\zeta = 2\pi i$$

Hence

$$f(z) \sim (-2\pi i a_1 + b_0) e^{-ik_0 z} \text{ as } x \rightarrow -\infty \quad (5.24)$$

We now apply the radiation condition (5.3) to f_c and f_s , adding appropriate subscripts to the coefficients:

$$f_c' + k_0 f_s' \sim -ik_0(b_{c0}' + ib_{c0}'') + k_0(b_{s0}' + ib_{s0}'') = 0 \quad \text{as } x \rightarrow +\infty$$

$$f_c' - k_0 f_s' \sim -ik_0(-2\pi ia_{c1} + b_{c0}' + ib_{c0}'')$$

$$-k_0(-2\pi ia_{s1} + b_{s0}' + ib_{s0}'') = 0 \quad \text{as } x \rightarrow -\infty$$

These equations yield

$$b_{s0}' = -b_{c0}'' = -\pi a_{c1}, \quad b_{c0}' = b_{s0}'' = \pi a_{s1} \quad (5.25)$$

Let us now go back to (5.21) and substitute this form for f_1 :

$$f_c = a_{c1} \left[- \int_0^\infty \frac{e^{-ikz}}{k-k_0} dk - \pi i e^{-ik_0 z} \right] + \pi(a_{s1} + ia_{c1})e^{-ik_0 z} \\ + \sum_{n=2}^\infty b_{cn} m_n,$$

$$f_s = a_{s1} \left[- \int_0^\infty \frac{e^{-ikz}}{k-k_0} dk - \pi i e^{-ik_0 z} \right] + \pi(-a_{c1} + ia_{s1})e^{-ik_0 z} \\ + \sum_{n=2}^\infty b_{sn} m_n$$

These simplify to the following forms:

$$\begin{aligned}
f_c &= -a_{c1} \int_0^\infty \frac{e^{-ikz}}{k-k_0} dk + \pi a_{s1} e^{-ik_0 z} + \sum_{n=2}^\infty b_{cn} m_n \\
f_s &= -a_{s1} \int_0^\infty \frac{e^{-ikz}}{k-k_0} dk - \pi a_{c1} e^{-ik_0 z} + \sum_{n=2}^\infty b_{sn} m_n
\end{aligned}
\tag{5.26}$$

Before returning to the problem of satisfying the boundary condition on the body, we make one more digression, namely, to explain why this is known as the method of multipoles. Suppose that we wish to construct a function behaving like a source of strength $Q \cos \sigma t$ and vortex of intensity $\Gamma \cos \sigma t$ at the point c in the lower half-plane, i.e., like

$$\frac{\Gamma+iQ}{2\pi i} \log (z-c) \cos \sigma t$$

and also satisfying the free-surface and radiation conditions and vanishing as $y \rightarrow -\infty$. This problem can be solved (e.g., by the reduction method) and we find

$$\begin{aligned}
G(z, c) &= \left[\frac{\Gamma+iQ}{2\pi i} \log (z-c) + \frac{\Gamma-iQ}{2\pi i} \log (z-\bar{c}) \right. \\
&\quad \left. - i \frac{\Gamma-iQ}{\pi} \int \frac{e^{-ik(z-\bar{c})}}{k-k_0} dk \right] \cos \sigma t \\
&\quad - i(\Gamma-iQ) e^{-ik_0(z-\bar{c})} \sin \sigma t \\
&= G_c \cos \sigma t + G_s \sin \sigma t
\end{aligned}
\tag{5.27}$$

The method of multipoles, as applied to the problem we have been considering, would consist in assuming that the solution can be expressed as a sum of G and its derivatives evaluated at $c = 0$, i.e., that

$$\begin{aligned}
f_c &= a_0 G_c(z, 0) + a_1 G_c'(z, 0) + a_2 G_c''(z, 0) + \dots, \quad a_n \text{ real} \\
f_s &= a_0 G_s(z, 0)
\end{aligned}
\tag{5.28}$$

Since there is no circulation in the problem we are considering, we may set $\Gamma=0$. Furthermore, there is no loss in generality in setting $Q=1$ since each summand is multiplied by a real coefficient to be determined later. If we also set $c=0$, (5.27) then takes the following form:

$$G_c(z,0) = -\frac{1}{\pi} \oint \frac{e^{-ikz}}{k-k_0} dk$$

$$G_s(z,0) = -e^{-ik_0 z} \quad (5.29)$$

Now if we examine the terms of (5.26) before the summation, we see that by setting $a_{sl}=0$, $b_{sn}=0$, we have exactly the same series as (5.28). Although (5.26) may appear to be more general because of the presence of the terms with a_{sl} , and b_{sn} , this is only an apparent generality, an appropriate shift of the time axis can achieve $a_{sl}=b_{sn}=0$. Hence the method we have developed is equivalent to the method of multipoles.

Next we must satisfy the boundary condition (5.7) by proper choice of the coefficients a_{cl} , a_{sl} , b_{cn} , and b_{sn} . For the case to which we are presently restricting ourselves, that of the semisubmerged circular cylinder, this condition may be given a slightly simpler form. In order to exploit symmetry, we introduce the angle γ shown in Figure 32.

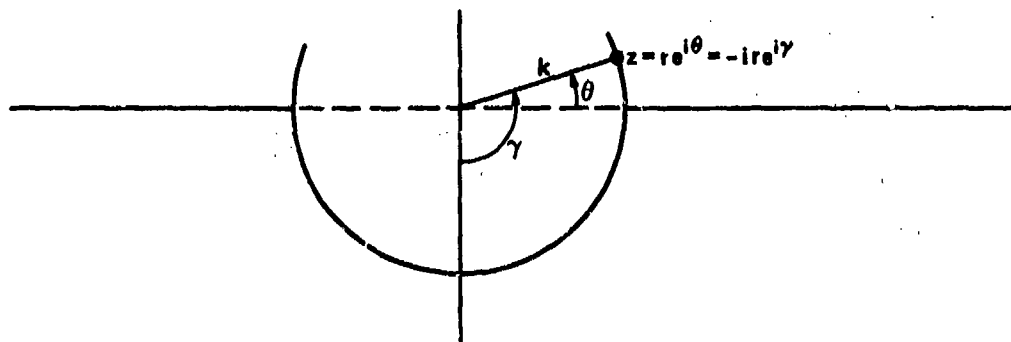


Figure 32 - Definition of Angle γ

As in (4.13), we shall write

$$V_n(x, y, t) = V_{nc} \cos \sigma t + V_{ns} \sin \sigma t \quad (5.30)$$

Then (5.7) takes the form

$$\begin{aligned} \phi_{cn} &= -\phi_{cx} \sin \gamma + \phi_{cy} \cos \gamma = V_{nc}(\gamma) \\ \phi_{sn} &= -\phi_{sx} \sin \gamma + \phi_{sy} \cos \gamma = V_{ns}(\gamma) \end{aligned} \quad (5.31)$$

It will be sufficient to discuss one of these although, as we shall see, the two must be solved together. From the solution (5.26) we may compute

$$f_c' = +ia_{c1} \int_0^\infty \frac{e^{-1kz}}{k-k_0} k dk - i\pi a_{s1} k_0 e^{-ik_0 z} + \sum_{n=2}^\infty b_{cn} m_n' \quad (5.32)$$

From (5.18)

$$m_n'(z) = -(n+1) \frac{1}{z^{n+2}} - ik_0 \frac{1}{z^{n+1}} = -(n+1)m_{n+1}(z) \quad (5.33)$$

Into (5.32) and (5.33), we substitute $z = -ire^{i\gamma}$ and separate real and imaginary parts. This somewhat tedious computation yields (perhaps) the following:

$$\begin{aligned} \phi_{cx} &= a_{c1} \int_0^\infty \frac{e^{-kr \cos \gamma} \sin(kr \sin \gamma)}{k-k_0} k dk \\ &\quad - \pi a_{s1} k_0 e^{-k_0 r \cos \gamma} \sin(k_0 r \sin \gamma) \\ &\quad + \sum_{p=1}^\infty b_{c,2p} \frac{2p+1}{r^{2p+2}} (-1)^p \left[\cos(2p+2)\gamma - \frac{k_0 r}{2p+1} \sin(2p+1)\gamma \right] \\ &\quad + \sum_{q=1}^\infty b_{c,2q+1} \frac{2q+2}{r^{2q+3}} (-1)^{q+1} \left[\sin(2q+3)\gamma + \frac{k_0 r}{2q+2} \cos(2q+2)\gamma \right] \end{aligned} \quad (5.34)$$

$$\begin{aligned}
\phi_{cy} = & -a_{cl} \int_0^{\infty} \frac{e^{-kr \cos \gamma} \cos(kr \sin \gamma)}{k - k_0} k dk \\
& + \pi a_{sl} e^{-k_0 r \cos \gamma} \cos(k_0 r \sin \gamma) \\
& + \sum_{p=1}^{\infty} b_{c,2p} \frac{2p+1}{r^{2p+2}} (-1)^{p+1} \left[\sin(2p+2)\gamma + \frac{k_0 r}{2p+1} \cos(2p+1)\gamma \right] \\
& + \sum_{q=1}^{\infty} b_{c,2q+1} \frac{2q+2}{r^{2q+3}} (-1)^{q+1} \left[\cos(2q+3)\gamma - \frac{k_0 r}{2q+2} \sin(2q+2)\gamma \right]
\end{aligned}$$

It seems reasonable now to skip over some details and to observe that (5.31) takes the following form:

$$\begin{aligned}
a_{cl} L_c(\gamma) + a_{sl} M_c(\gamma) + \sum_{n=2}^{\infty} b_{cn} N_c(\gamma) &= V_{nc}(\gamma) \\
a_{cl} L_s(\gamma) + a_{sl} M_s(\gamma) + \sum_{n=2}^{\infty} b_{sn} N_s(\gamma) &= V_{ns}(\gamma)
\end{aligned} \tag{5.35}$$

There are evidently an infinite number of unknown coefficients, but also, of course, an infinite number of values of γ where these equations must be satisfied. We shall not discuss the methods that have been used to solve approximately these equations by truncation. Unfortunately, the various functions have no orthogonality properties.

We must now consider the question of what to do if the cross section of the body is not a semicircle, for as we have seen earlier (see page 97), we have no reason to believe that the circle of convergence of the function

$g(z)$ in (5.10) will be inside the body boundary (plus its reflection). What we do in this case is to make a preliminary mapping of the exterior of the body plus its reflection onto the exterior of a circle. However, the mapping chosen imposes a restraint upon the body: it must intersect the free surface perpendicularly.

Let ζ be the plane of the circle and z the physical plane containing the body in question (see Figure 33).

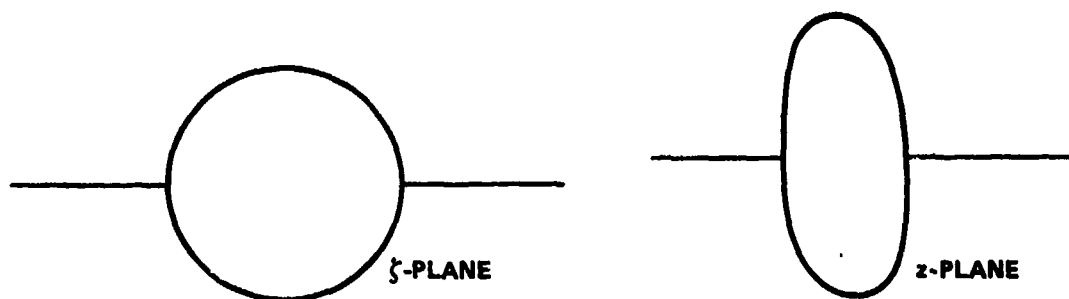


Figure 33 - ζ - and z -Planes

Then it is known that we can map the exterior of the circle onto the exterior of the body and the ξ -axis into the x -axis by a mapping of the form

$$z = \zeta + \sum_{m=1}^{\infty} \frac{C_m}{\zeta^m}, \quad C_m \text{ real} \quad (5.36)$$

Let us suppose that $f(z)$ is one of f_c or f_g . Then

$$f(z) = f(z(\zeta)) \equiv F(\zeta) \quad (5.37)$$

defines an analytic function in the ζ -plane that can be considered as a velocity potential there. What happens to the free-surface condition? Since $F'(\zeta) = f'(z) z'(\zeta)$, we find

$$f'(z) + ik_0 f(z) = \frac{F'(\zeta)}{z'(\zeta)} + ik_0 F(\zeta)$$

Hence condition (5.2) becomes

$$I_m \left\{ \frac{F'(\xi+10)}{z'(\xi+10)} + ik_0 F(\xi+10) \right\} = 0$$

or, since

$$z'(\xi+10) = 1 - \sum_{m=1}^{\infty} mC_m \xi^{-m-1}$$

is real,

$$I_m \left\{ F'(\xi+10) + ik_0 \left[1 - \sum_{m=1}^{\infty} mC_m \xi^{-m-1} \right] F(\xi+10) \right\} = 0 \quad (5.38)$$

This is the boundary condition that must be satisfied on the real axis outside the circle.

We shall now proceed similarly to the earlier case. Define

$$G(\zeta) = F'(\zeta) + ik_0 \left[1 - \sum_{m=1}^{\infty} \frac{mC_m}{\zeta^{m+1}} \right] F(\zeta) \quad (5.39)$$

This function can be extended by reflection into the whole complex plane and furthermore can be shown by easy arguments to be bounded outside some circle $|\zeta| = a$ containing the mapping circle. As before, we find

$$G(\zeta) = \sum_{n=1}^{\infty} \frac{a_n}{\zeta^n}, \quad a_n \text{ real}, \quad |\zeta| \geq a \quad (5.40)$$

But then, since $G(\zeta)$ must be analytic right up to the boundary of the mapping circle, the Laurent series (5.40) must converge right up to and including this circle.

We are now left with the problem of integrating the differential equation

$$F'(\zeta) + ik_0 \left[1 - \sum_{m=1}^{\infty} \frac{mC_m}{\zeta^{m+1}} \right] F(\zeta) = \sum_{n=1}^{\infty} \frac{a_n}{\zeta^n} \quad (5.41)$$

The integral can be found in textbooks on differential equations (e.g., Kamke, "Differentialgleichungen: Lösungsmethoden und Lösungen," p. 16) and may be written as follows:

$$F(\zeta) = \exp\{-ik_0[\zeta + \sum_m C_m \zeta^{-m}]\} \left\{ B + \sum_n a_n \int_{\infty}^{\zeta} \zeta^{-n} \exp\{ik_0[\zeta + \sum_m C_m \zeta^{-m}]\} d\zeta \right\} \quad (5.42)$$

where the path of integration is below the mapping circle and B is complex.

Let us define

$$F_n(\zeta) = \exp\{-ik_0[\zeta + \sum_m C_m \zeta^{-m}]\} \int_{\infty}^{\zeta} \zeta^{-n} \exp\{ik_0[\zeta + \sum_m C_m \zeta^{-m}]\} d\zeta \quad (5.43)$$

Then (5.42) may be written in the form

$$F(\zeta) = B \exp\{-ik_0[\zeta + \sum_m C_m \zeta^{-m}]\} + \sum_{n=1}^{\infty} a_n F_n(\zeta) \quad (5.44)$$

If we integrate F_n once by parts for $n \geq 2$, we find

$$\begin{aligned} F_n &= \frac{\zeta^{-n+1}}{-n+1} + \frac{1}{n-1} \exp\{-ik_0[\zeta + \sum_m C_m \zeta^{-m}]\} \times \\ &\quad \int_{\infty}^{\zeta} ik_0 \left[1 - \sum \frac{mC_m}{\zeta^{m+1}} \right] \frac{1}{\zeta^{n-1}} \exp\{ik_0[\zeta + \sum_m C_m \zeta^{-m}]\} d\zeta \\ &= -\frac{1}{n-1} \frac{1}{\zeta^{n-1}} + \frac{1}{n-1} ik_0 F_{n-1} - \frac{ik_0}{n-1} \sum_{m=1}^{\infty} mC_m F_{m+n} \end{aligned}$$

Let us replace n by n+1 and rearrange the result as follows:

$$F_n(\zeta) - \frac{n}{ik_0} F_{n+1} - \sum_{m=1}^{\infty} mC_m F_{m+n+1} = \frac{1}{ik_0} \frac{1}{\zeta^n}, \quad (5.45)$$

$$n = 1, 2, 3, \dots$$

If possible, we should like to do something like we did on pages 99-100, i.e., to replace the awkward integrals F_2, F_3, \dots by polynomials, something analogous to the $m_n(z)$ of (5.18). Equation (5.45) is the analog of (5.13). However, to manipulate terms as we did in (5.16) and (5.17) would seem to require a remarkable insight. Let us write out (5.45) for $n = 2, 3, \dots$ for several terms:

$$\begin{aligned}
 F_2 - \frac{2}{ik_0} F_3 - C_1 F_4 - 2C_2 F_5 - 3C_3 F_6 - \dots &= \frac{1}{ik_0} \frac{1}{\zeta^2} \\
 F_3 - \frac{3}{ik_0} F_4 - C_1 F_5 - 2C_2 F_6 - \dots &= \frac{1}{ik_0} \frac{1}{\zeta^3} \\
 F_4 - \frac{4}{ik_0} F_5 - C_1 F_6 - \dots &= \frac{1}{ik_0} \frac{1}{\zeta^4} \\
 F_5 - \frac{5}{ik_0} F_6 - \dots &= \frac{1}{ik_0} \frac{1}{\zeta^5} \\
 \vdots & \\
 \vdots &
 \end{aligned}
 \tag{5.46}$$

It seems evident that F_n must be expressible as a series starting with ζ^{-n} :

$$F_n = \frac{1}{\zeta^n} \sum_{k=0}^{\infty} \frac{B_k^{(n)}}{\zeta^k} \tag{5.47}$$

If we substitute this expression into the defining equation for F_n :

$$* \quad F_n' + ik_0 \left[1 - \sum_{m=1}^{\infty} \frac{mC_m}{\zeta^{m+1}} \right] F_n = \frac{1}{\zeta^n}$$

it is not difficult to establish the following:

$$B_0^{(n)} = \frac{1}{ik_0}, \quad B_1^{(n)} = \frac{-n}{k_0^2}, \quad B_2^{(n)} = \frac{n+1}{ik_0} B_1^{(n)} + C_1 B_0^{(n)}$$

and in general that

$$B_{q+1}^{(n)} = \frac{n+q}{ik_0} B_q^{(n)} + \sum_{r=0}^{q-1} (q-r) c_{q-r} B_r^{(n)}, \quad q \geq 1 \quad (5.48)$$

Although we could proceed with this expression for F_n , it would require careful attention to detail in separating real and imaginary parts in the coefficients $B_q^{(n)}$, $q \geq 2$.

In order to avoid this difficulty, it has been customary to proceed somewhat differently. Instead of the sum $\sum a_n F_n$ in (5.44), we shall try to replace it by another sum $\sum d_n M_n$, where the M_n will be defined below. We shall again suppose that M_n has the form

$$M_n = \frac{1}{\zeta^n} \sum_{k=0}^{\infty} \frac{b_k^{(n)}}{\zeta^k} \quad (5.49)$$

but we shall begin by imposing only the condition

$$\text{Im} \left\{ M_n' + ik_0 \left[1 - \sum_{m=1}^{\infty} \frac{m c_m}{\zeta^{m+1}} \right] M_n \right\} = 0 \quad \text{when } n = 0 \quad (5.50)$$

Of course, F_n satisfies this condition also, so that we must expect to impose some additional conditions. Substituting (5.49) into (5.50), we obtain the following equation (we temporarily drop the superscript n):

$$\begin{aligned} & \text{Im} \left\{ ik_0 b_0 + (ik_0 b_1 - nb_0) \zeta^{-1} + \right. \\ & \left. + \sum_{k=1}^{\infty} \left[ik_0 b_{k+1} - (k+n)b_k - ik_0 \sum_{r=0}^{k-1} (k-r) c_{k-r} b_r \right] \zeta^{-k-1} \right\} \end{aligned} \quad (5.51)$$

If we now set $b_k = h_k' + ib_k''$ and $\zeta = \xi + i0$, the coefficients of each power of ξ yield the following equations:

$$\begin{aligned}
b_0' &= 0 \\
b_1' &= \frac{n}{k_0} b_0'' \\
b_2' &= \frac{n+1}{k_0} b_1'' \\
b_3' &= \frac{n+2}{k_0} b_2'' + C_1 \\
b_4' &= \frac{n+3}{k_0} b_3'' + 2C_2 + C_1 b_2' \\
&\vdots \\
b_{k+1}' &= \frac{n+k}{k_0} b_k'' + \sum_{r=0}^{k-1} (k-r) C_{k-r} b_r'
\end{aligned} \tag{5.52}$$

Evidently, for each value of k , b_k'' is not determined, but b_k' is determined as a linear function of the preceding b_0', \dots, b_{k-2}' , and b_{k-1}'' . We can choose the b_k'' as we please. We shall do this in such a way as to make $b_k' = 0$ for $k \geq 2$. This can be accomplished by choosing

$$b_0'' = \frac{k_0}{n}, b_1'' = 0, b_2'' = -k_0 \frac{C_1}{n+2}, \dots, b_k'' = -k_0 \frac{(k-1)C_{k-1}}{n+k}, \dots \tag{5.53}$$

The only nonzero real part is $b_1' = 1$. The series (5.49) then has the following form:

$$M_n(\zeta) = \frac{ik_0}{n} \frac{1}{\zeta^n} + \frac{1}{\zeta^{n+1}} - ik_0 \sum_{k=1}^{\infty} \frac{kC_k}{n+k+1} \frac{1}{\zeta^{n+k+1}} \tag{5.54}$$

The M_n may be considered as the analogs of the m_n of (5.19).

We have not yet shown that $\sum_{n=1}^{\infty} a_n F_n$ can be replaced by $a_1 F_1 + \sum_{n=2}^{\infty} d_n M_n$ with d_n real. With some further manipulation, we can establish the following relation between F_n and M_n :

$$M_n = -\frac{k_0^2}{n} F_n + \left[k_0^2 \left(\frac{1}{n} + \frac{1}{n+2} \right) - n-1 \right] F_{n+2} + 2k_0^2 \left(\frac{1}{n} + \frac{1}{n+3} \right) F_{n+3} \\ + k_0^2 \sum_{r=4}^{\infty} \left\{ (r-1) \left(\frac{1}{n} + \frac{1}{n+r} \right) - \sum_{k=1}^{r-3} \frac{k(r-k-2)}{n+k+1} C_k C_{r-k-2} \right\} \quad (5.55)$$

Since the coefficients on the right-hand side are all real, it is evident

that a sum $\sum_{n=2}^{\infty} d_n M_n$ with d_n real is equivalent to a sum $\sum_{n=2}^{\infty} a_n F_n$ with a_n real. Since the coefficients a_n have not yet been determined, it is not necessary to know the relation between the $\{a_n\}$ and the $\{d_n\}$. Instead, we shall determine the d_n directly.

Equation (5.44) for $F(\zeta)$ now takes the form

$$F(\zeta) = B \exp\{-ik_0[\zeta + \sum_m \zeta_m^{-m}]\} + a_1 F_1 + \sum_{n=2}^{\infty} d_n M_n, \quad d_n \text{ real} \quad (5.56)$$

The radiation condition and the boundary condition on the body remain to be considered. For this purpose there may be some gain in replacing $F_1(\zeta)$ by $f_1(z(\zeta))$ where f_1 is defined by (5.12) with $n=1$. This is not, of course, the same function as $F_1(\zeta)$, but it can be used in (5.56), as we shall see below:

$$\int_{\infty}^z \frac{e^{ik_0 \lambda}}{z} dz = \int_{\infty}^z \frac{\exp ik_0[\zeta + \sum_m \zeta_m^{-m}]}{\zeta + \sum_m \zeta_m^{-m}} \left[1 - \sum \frac{m \zeta_m}{\zeta^{n+1}} \right] d\zeta$$

$$= \int_{\infty}^{\zeta} \frac{\exp ik_0 [\zeta + \sum C_m \zeta^{-m}]}{\zeta} \frac{1 - \sum \frac{m C_m}{\zeta^{m+1}}}{1 + \sum \frac{C_m}{\zeta^{m+1}}} d\zeta$$

$$= \int_{\infty}^{\zeta} \frac{\exp ik_0 [\zeta + \sum C_m \zeta^{-m}]}{\zeta} \left[1 + \sum_{n=1}^{\infty} \frac{e_n}{\zeta^n} \right] d\zeta$$

where e_n is real. Evidently the difference between $f_1(z(\zeta))$ and $F_1(\zeta)$ will be a sum $\sum_{n=2}^{\infty} e_n F_n$ which, in turn, can be replaced by a sum of M_n . Hence we may replace (5.53) by

$$F(\zeta) = B \exp\{-ik_0 [\zeta + \sum C_m \zeta^{-m}]\} + a_1 f_1(z(\zeta)) +$$

$$+ \sum_{n=2}^{\infty} d_n M_n, \quad d_n \text{ real} \quad (5.57)$$

where these d_n are different in general from those of (5.56).

With this change, we may take over completely our earlier relations on page 104 that were derived to satisfy the radiation condition, for the M_n play no role in this condition.

In order to satisfy the condition (5.7) on the body in the physical plane, we must establish a relation between this condition and one in the ζ -plane. Let us first rewrite (5.7) for the case where the parameter is no longer arc length. Let it be p . Then (5.7) becomes

$$\operatorname{Im} f'(z(p)) \frac{z'(p)}{|z'(p)|} = -v_n \quad (5.58)$$

Let p now be a parameter in the ζ -plane describing the mapping circle: $\zeta(p)$. Then the body in the physical plane is described by

$$z(p) = z(\zeta(p))$$

and

$$z'(p) = z'(\zeta(p)) \zeta'(p)$$

Since

$$f'(z) z'(\zeta) = F'(\zeta)$$

we find

$$f'(z(p)) z'(p) = f'(z) z'(\zeta) \zeta'(p) = F'(\zeta) \zeta'(p)$$

condition (5.59) then becomes the following in the ζ -plane:

$$\operatorname{Im} \frac{F'(\zeta(p)) \zeta'(p)}{|z'(\zeta(p)) \zeta'(p)|} = -v_n \quad (5.59)$$

To develop the analogs of (5.35) would involve us in more details than seem appropriate to carry through in these lectures. We terminate the discussion of this method with the remark that it is possible to develop this into a computational technique for bodies intersecting the free surface perpendicularly, as has been shown by Tasai (1959) and Porter (1960).

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